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PAUL KLAPPER AND RALEIGH SCHORLING

THE TEACHING OF ARITHMETIC

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BY

PAUL KLAPPER, Ph.D.

DEAN OF THE SCHOOL OF EDUCATION, COLLEGE OF THE CITY OF NEW YORK;
AUTHOR OF "CONTEMPORARY EDUCATION, ITS PRINCIPLES
AND PRACTICE," "THE TEACHING OF HISTORY,"
"TEACHING CHILDREN TO READ," ETC.

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PREFACE

THE author set out to revise an earlier book of the same title, but realised that it would be the part of wisdom to attempt a new book with only incidental reference to his first text. What is here offered is not a revision, in the usual sense, but, we believe, a richer exposition of methodology of teaching. It is hoped that the reader will find this book more than a manual of method.

Three primary objectives have been kept in mind:

First, to arouse in the reader a keen discontent with teaching procedure based on personal opinion. Throughout the book, the student is given a summary of the researches related to the specific subject under discussion and is encouraged to determine whether conclusive data have as yet been deduced to justify a final decision. Thus, he is gradually introduced to many of the experimental studies in the field of elementary mathematics and to the techniques of scientific method in modern education.

Second, to supply for every teaching technique that is advocated a justification in terms of accredited principles of educational psychology. Principles and practices are intimately associated in the belief that principles without practice are empty verbalisms, and that practice without principles is blind procedure.

Third, to lead the student to realize that the emphasis must be placed on learning rather than on teaching arithmetic. Each fundamental operation is, therefore, reduced to its important unit skills; modes of administering adequate diagnostic tests and of formulating individualized, remedial practice are stressed throughout. The teacher is helped to study the pupil, to diagnose his weaknesses, and to devise corrective measures. The author tried to be ever mindful of actual school conditions and of the needs of the professionally minded teacher who makes teaching a benevolent guidance rather than an intellectual domination. For every basic topic or process, the teacher is helped to formulate a genuinely motivated approach, a system of gradation, a method of rationalization, and, finally, functional practice and vitalized applications to life about us.

The Teaching of Arithmetic addresses itself to teachers and to prospective teachers as well as to supervisors, to teachers of primary as well as of intermediate grades, because the author believes that one integrated body of facts ought to be presented to all who are educating children. We cannot set apart a division of the subject that should concern the supervisor and not the teacher. Both seek the same ends, albeit in different ways. The supervisor who is not expert in methodology cannot supervise helpfully; the teacher who is a stranger to the objectives, the standards, and the techniques of her supervisor may find herself, unwittingly, working at cross purposes. If supervision is to cease being a control from above and is to become a coöperative enterprise, then the study of methodology and supervisory problems should not be segregated in separate books or in separate courses.

The author is grateful to the many teachers whose classwork has suggested much that is helpful in these pages. He is especially thankful to his coeditor of the Appleton Series in Special Methods, Professor Raleigh Schorling, for a very critical reading of the manuscript and for many constructive suggestions.

P. K.

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PART I

VALUES AND OBJECTIVES

CHAPTER I

WHAT CAN ARITHMETIC GIVE THE CHILD?

Need of formulating definite aims. In a sense, any teaching method may be good, and no method of teaching may be entirely adequate. Not the technique of the method, but the degree to which it achieves its goal determines its worth as a teaching procedure. Our initial problem, therefore, is to formulate a set of aims or objectives to be attained through the teaching of arithmetic. In terms of these, we shall judge the worth of courses of study, methods of teaching, and techniques of testing.

We must distinguish between the aims of a subject as a whole and the aims of a specific lesson in it. In the former sense the term *aim* is synonymous with *value*; in the latter, the term *aim* refers to mastery of a *specific skill* or comprehension of a definite process. Thus, a course of study in arithmetic sets up three aims to be achieved in the fifth and sixth grades—to increase accuracy, to encourage planning of procedure, and to reduce the checking of results to habit. Here clearly are three values which children are to attain through learning arithmetic in two school years. The aim of a particular lesson in multiplication of fractions is to lead pupils to understand why multiplying by a fraction that is less than one always produces a result smaller than the multiplicand; the aim of a lesson in division may be to facilitate the finding of the successive quotient figures, so that, in dividing 1248 by 18, the child will not make a wild guess with 18 into 124 but will think, "20 into 120, write 6." In the first of these two lessons one aim is to lead to understanding of a particular process; in the second, to develop control of a technique of operation. Throughout this chapter we shall consider the aims of the subject of arithmetic as a whole. *Aim*, therefore, is used in the sense of *value*.

Teachers of arithmetic have long recognized that this subject held a dominating place in elementary education and have evolved a series of values to justify its position. These values may be classified as:

1. The social values
2. The traditional values, which include
   - *a.* The disciplinary values
   - *b.* The pleasure values
   - *c.* The cultural values
   - *d.* The preparatory values

The social values of arithmetic. The utilitarian values of arithmetic have long served to justify its traditional importance in elementary education. The simplest routine of daily life demands a mastery of certain number facts and number processes. This need is obvious in such everyday social acts as making simple purchases, making change, counting money, telling time, using a calendar, laying out courts and play spaces, keeping a household or a personal budget, reading a thermometer, keeping a personal account, making a record of one's school progress, reading telephone, house, and automobile numbers—in a word, most of the activities that make up daily life.

In the schools of Springfield, Massachusetts, a record was made of the number experiences noted during one day's activities in a kindergarten. The list, surprisingly long and inclusive, is given below:

Measurement and comparison. Measured legs for doll's chair, placed them at equal distances, right height for doll. Doll's table made tall enough for doll's chair to slip under. Table made in proportion to other furniture. Bed made to fit small doll. Length of legs for furniture measured so as to stand straight. Used ruler as guide for saw. Girls measured paper dolls for dresses. Group made doll's house with blocks for sand-table project, measured distance to be covered, estimated how many blocks required. Made trees out of sticks and green paper, used different lengths so that trees would not be the same height.

United States money. Lunch money; children asked to tell how much they bought. Stamp money. How much bought.

Counting. Number of boys and girls present. Animal crackers (each child helped himself to a given number). Boys built a barn, counted animals to fill the stalls. Each child asked to take two pieces of paper. Five pictures—to choose one. One boy took "the fourth from the end."[1]

But membership in modern civilized society requires more than the bare outline of arithmetical knowledge here suggested. We must have sufficient skill and understanding of quantitative relations to decide whether we can afford to make certain purchases, whether installment buying is safe under certain circumstances, whether to buy or rent a home. We must know how taxes are levied, how the return on an investment is computed, and how an average is obtained—briefly, the common business practices of social life.

The study of arithmetic must lead to a fuller understanding of many of life's problems from a quantitative point of view. We study current problems from economic, social, artistic, and ethical aspects. Similarly we may interpret these questions from a statistical or a quantitative viewpoint. To know that the United States produces cotton, grain, and lumber is to possess colorless geographical information; to realize that the United States produces three-fifths of the cotton, one-third of the grain, and one-third of the lumber of the world adds meaning and vital significance to these facts. To find that the attendance in a given school increased in one year from 2005 to 2995 may give no adequate appreciation of its growth, but to know that the increase was almost one-half, or about 50 per cent of the original attendance, stirs the mind to a vivid understanding of such unusual development. Not many people in a given trade seemed to understand the statement: "True, the cost of living has increased 80 per cent during these ten years, but wages have not stood still; the average wage in

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[1] "Report on Arithmetic" in *Fourth Yearbook* of the Department of Superintendence of the National Education Association, Washington, D. C., February, 1926, pp. 192-193.

[illegible] has gone up from $1480 to a little over $2000 a year. Not until a keen-witted member of the group reduced the figures to a common denominator were they aroused to indignation. For now the workers understood the full force of the discrepancy—cost of living increased 80 per cent, wages, [illegible] per cent. Arithmetical interpretations add new views to [illegible] and thus enrich our understanding of certain social [illegible]

[illegible] *of arithmetic exaggerated.* In the past [illegible] of arithmetic were often exaggerated [illegible] bounds. Arithmetic knowledge seemed [illegible] in the school curriculum. Children were promoted [illegible] according to their proficiency in this one subject. [illegible] this chapter we shall analyze the relatively limited [illegible] of arithmetic knowledge that the average intelligent [illegible] needs and we shall indicate the field of experience that [illegible]. Comparison of the arithmetic skills em[illegible] by the bookkeeper with the set of skills that most [illegible] of study seek to develop gives one a quick appraisal [illegible] extent to which we have overburdened the content of [illegible]. One considers whether courses of study are [illegible] that it is necessary for all children to learn [illegible] places, to manipulate fractions with such [illegible] 39, 28, 21, and 42, to divide by 80 809, and [illegible] the area of a triangular pyramid. To go beyond the [illegible] of social usefulness may involve waste of time and [illegible] may needlessly discourage those possessed of [illegible] intelligence.

[illegible] *on social utility?* It is no simple matter to [illegible] a standard in terms of which one may decide [illegible] given topic in arithmetic is sufficiently useful [illegible] to warrant its inclusion in a course of study. The difficulty arises from one of two sources:

First, the standard must not express a personal opinion; [illegible] the contrary, it must be objectively derived. Too long have [illegible] of textbooks, framers of courses, and supervisors decided subjectively what, in the light of their experience or their judgment, merits inclusion in the school curriculum. Unless we ascertain, by analysing typical commercial, industrial, domestic, and recreational situations, the arithmetic knowledge and skill actually needed, courses of study will continue to express personal whim or the prevailing but none-the-less temporary opinion of professional educators. A good brief can be made out for the inclusion of many topics that appear useless. What an imposing array of supporting reasons can be presented to justify the teaching of number puzzles! Yet common sense overrides them all and puzzles are, generally speaking, excluded from grade teaching.

A second source of difficulty arises from the fact that specific vocations, which have been studied to discover what arithmetical skills are needed, yield conflicting data and point to seemingly irreconcilable conclusions. An analysis of the arithmetical needs of clerks in a department store seems to indicate that we should teach little beyond the four processes, the taking of such fractional parts as one-half, one-third, one-fourth, one-eighth, and subtraction of reasonable amounts from five, ten, and twenty dollars. But clerks employed in the catalog-publication department of wholesale houses seem to need that much condemned "Case III of Discount." On the other hand it is difficult indeed to find vocations that justify the time and energy spent by the school in developing expertness in multiplication and division by large integers.

It was found that the average adult, in a group of 331 selected from the membership of parent-teacher organizations, rotary clubs, and similar groups, exceeds the eighth-grade norm in certain processes, but falls below this norm in others.[2] *The Cleveland Survey Test in Arithmetic* was used and yielded ground for the following conclusions:[3]

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[2] Of course adults are not in the same condition of training as children in the eighth grade; with a little practice they can probably make an appreciable improvement. Nor is it probable that the degree of skill possessed by children of Grade 8 will remain constant.

[3] Carleton W. Washburne, "Social Practices in Arithmetic Fundamentals," *Elementary School Journal*, Vol. 27, September, 1926, pp. 60-66.

It seems reasonable to reduce present-day emphasis on fractions and on multiplication and division of integers, satisfying ourselves with an understanding of the processes, and allowing life's experiences to supply the practice that increases skill. In addition and subtraction of integers we must raise our standards; in fact, we can hardly overtrain in these. "Yet the skill in them which we do not have time to impart will be imparted, in many cases, by life itself."[4]

The studies made to find the arithmetical needs in different occupations reveal the needs in particular vocations, but fail utterly to indicate a common denominator of needs, that is, the minimum arithmetical knowledge and skills required by most callings. It is clear that in any one vocation relatively little is required when we compare its arithmetic needs with the content of the best elementary-school textbooks in this subject. The researches also reveal that current business practice tends to simplify arithmetical operations by methods of marking goods, fixing the content of packages, and supplying computing scales and tables.[5]

The disciplinary values of arithmetic. Although arithmetic is commonly regarded as a tool subject, many serious teachers see it as a subject rich in disciplinary values. The permanent effects of any experience upon mind and behavior are its disciplinary values. These may be either desirable or undesirable. Many writers think of the former when they employ the term *disciplinary values.*

"Learn arithmetic," the child was advised, "for it will teach you to analyze a situation, to come to a decision, to check your thinking and its result, to perceive relationships, to concentrate, to be accurate, and to perform written work systematically and neatly. You may forget the number of feet in a mile, how to extract the square root, how to find the least common denominator, but you will acquire for all time invaluable habits of thought, concentration, and accuracy." Naturally, the disciplinary coefficient ascribed by these teachers to arithmetic is unusually high, and accounts for the controlling place it held so long.

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[4] *Ibid.*, p. 66.

[5] A summary of the investigations of arithmetical needs of different vocations will be found in the *Third Yearbook* of the Department of Superintendence of the National Education Association (Washington, D. C., 1925), pp. 40-68. This volume also contains an excellent report on "Curriculum Problems and Their Scientific Solution," pp. 23-110.

Extreme disciplinists have an unshakable faith that all these desirable habits of mind acquired in arithmetic will be transferred to other experiences where accuracy, concentration, and keen thinking are essential. Do we not meet professors of mathematics who insist that all who are preparing for professional and technical callings should study mathematics, even though its body of facts and skills may not enter into their future daily tasks? To quote:

> Are you to be a lawyer? How can you learn to analyze a complicated legal case, if you cannot learn to analyze a simple proposition of geometry? Are you a student of history? How can you determine the influence of Napoleon on the world's development if you are incapable of determining the influence of a coefficient in a simple relation of algebra? Are you a linguist? How will you translate a masterpiece, with its myriad shades of meaning, from one language into another, if you cannot learn to translate a trifling "reading problem" into the corresponding mathematical symbols? Are you to be a physician? How will you diagnose and eliminate a disease, with its complicated, ambiguous, and obscure symptoms, if you lack the faculties needed to diagnose and eliminate the unknown quantity out of an elementary equation?
>
> Mathematics typifies a method of reasoning . . . all life problems must undergo analysis before a solution can be obtained. . . . To be capable of handling a situation is power, regardless of occupation or social position. The study of mathematics is one of the most effective means to such an end.

To these teachers the mind is a vast reservoir into which flow powers which will be drained off as occasion arises; they see the mind as a great storage battery that is charged with mental energy by mathematics. When thinking must be performed the battery uses the power that has been stored in it.

Limitations of the disciplinary view. The position of the disciplinists is untenable; the mathematical training of these very students has not saved them from slovenly thinking.

All subjects, properly taught, have disciplinary values, but we have little creditable evidence that the habits of mind developed in one of them can be carried over to all the others without serious loss. Only as the identical elements of content or method or attitude of two subjects are perceived and generalized is the power developed in one of them transferred to the other. Thinking developed in arithmetic will aid in learning algebra and in solving problems in mathematical physics, but it will leave the student no better prepared to select his vocation, to judge people, or to solve a difficulty in personal relations. In all of these situations he must think and think accurately, he must concentrate, and he must check his conclusions before acting upon them. But the thinking in arithmetic is radically different from the thinking required to make a vocational choice. In fact, the thinking is so different that, were language rich enough in symbols, we should be compelled to use different words to identify these decidedly different mental operations.

While the extreme doctrine of formal discipline has long been overthrown, the view persists that ability developed in one branch of arithmetic can be utilized in all branches of the same subject. Experimental studies show that ability in arithmetic is a sum total of abilities, a complex of powers; that proficiency in arithmetic must be resolved into proficiency in combinations, in copying accurately, in controlling many subprocesses such as carrying, borrowing, and placing; that frequently children are not proficient in all branches of arithmetic, but only in some of them. One investigator,[*] who set before himself the problem of ascertaining whether numerical accuracy, that is, accuracy in mere number manipulation, is transferred to accuracy in the reasoning problems of arithmetic, found that improvement in accuracy in pure computations seems to produce no increase in accuracy in solving reasoning problems in arithmetic. It has also been shown that numerical accuracy should be made an end in arithmetic, not because it aids in the solution of reasoning situations in arithmetic, but because it is necessary in commercial and industrial life. To conclude that no transfer of accuracy takes place would be going beyond the limits warranted by the facts, for the accuracy developed in pure numerical operations aids in the numerical calculations involved in the solution of any problem. Starch,[7] studying this problem of transfer of ability from one branch of arithmetic to another, concludes that such proficiency as seems to be transferred is due to a certain general ability which is developed, and which is capable of being used as much in the mechanical operations as in the reasoning problems.

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[*] W. H. Winch, "Accuracy in School Children: Does Improvement in Numerical Accuracy Transfer?" *Journal of Educational Psychology*, Vol. I (1910), pp. 262–71.

Reasons for traditional emphasis on the disciplinary value. The place of honor which the disciplinists have always accorded arithmetic is not due to mere blind continuance of custom. Arithmetic requires a rather high order of intelligence for its mastery. Those who were proficient in arithmetic were, of course, the intelligent pupils who were frequently proficient in most other subjects with the possible exception of drawing, music, and physical training.[8] Too frequently an unwarranted inference was drawn that the learning of arithmetic developed a surplus of mind, which the child could spend on his other studies. The devotees of arithmetic overlooked the fact that the child's mastery of arithmetic gave evidence of favorable initial intelligence. "Scores on arithmetic-fundamental tests," it was found, "correlated more highly with the problem scores than did any other measure except that secured by the verbal intelligence test."[9]

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[7] D. Starch, "Transfer of Training in Arithmetical Operations," *Journal of Educational Psychology*, Vol. 2, 1911, pp. 306–310.

[8] Moore finds that the correlation among the several parts of arithmetic is positive and fairly high. Buckingham is led to the surprising conclusion that the correlation between scores in arithmetic tests and in general intelligence is low. While these statements are not proved by these investigations, they are worthy of consideration. The character of the test may produce the kind of correlation Buckingham reports. See R. C. Moore, "The Psychology of Number: A Study of Numerical Ability," *Journal of Experimental Pedagogy*, Vol. 4, June, 1918, pp. 221–226. B. R. Buckingham, "Mathematical Ability as Related to General Intelligence," *School Science and Mathematics*, Vol 21, March, 1921, pp. 205–215.

Children in progressively higher grades can count more rapidly and more accurately, even though no attention is given to counting in the arithmetic lessons beyond the initial grades. Though none will challenge this observation, it cannot be used as a proof that there is considerable transfer of skill from one branch of arithmetic to another. We may account for such continued improvement in various ways: (1) The children are maturing and learn from experience. (2) There is unmistakable similarity in content and procedure among certain parts of arithmetic;[10] to possess skill in one of them insures a measure of control in the other. (3) New generalizations light up preceding facts and help the child to orientate himself in a new subject. Thus, the child who has acquired 2 × 12, 3 × 8, 4×6, 6×4, 8×3, 12×2, sees enough in 24 to help him in no small way in his later divisions. (4) Finally, one derives meaning more from the organization of an experience than from its discrete or constituent elements. Here we come close to Gestalt psychology which stresses the constellation, the pattern of the elements, rather than the mere existence of the elements. As the child matures he sees the organization of the number experiences, the patterns or arrangements of individual things that spell number, and thus he acquires a mastery of counting that no further specific drills can hope to give.

This cursory review of the data gathered by experimental psychologists casts serious doubt on the extravagant disciplinary hopes that were entertained for mathematics. We are told repeatedly that the study of mathematics inculcates the habit of neatness and system in work, develops a feeling for absolute accuracy, and cultivates a "reverence for truth, irrespective of sect, prejudice, and self-interest." But the writer who makes these promises for arithmetic fails to add the necessary qualifying expression, "in all matters dealing with arithmetical solutions." The student who does his arithmetical work with system and neatness may not carry his neatness and system beyond the paper that bears the solution of the problems. To J. W. A. Young, mathematics can inculcate an esthetic sense, for "it has a beauty of its own, a symmetry and proportion of results, a lack of superfluity, an exact adaptation of means to an end which can be found only in the works of the greatest beauty." When viewed as an expression of faith in one's specialty, this statement is comprehensible; as an expression of actual fact, it is entirely at odds with the results of dispassionate studies on the controlling values of arithmetic and mathematics.

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[9] R. L. Morton, "An Analysis of Errors in the Solution of Arithmetic Problems," *Educational Research Bulletin*, Ohio State University, Vol. 4, April 29, 1925, p. 188.

[10] Collar's study points to "a single specific mental factor operative in all kinds of arithmetical work." See, D. J. Collar, "A Statistical Survey of Mathematical Ability," *British Journal of Psychology*, Vol. 11, Part I, October, 1920, pp. 135–158.

Conclusion: Significance of modern conception of transfer for the teaching of arithmetic. The correct understanding of the disciplinary value of arithmetic is vital and far-reaching in its consequence.[11] If power developed in mathematics is transferred to all other experiences, then we need not concern ourselves with the task of including in arithmetic only such facts and processes as are used in actual social life. Any problem, however artificial, or any mode of solution, however archaic, will serve to inculcate certain mental powers. But a clear comprehension of the extent and the nature of transfer of abilities leads at once to a realization that arithmetic must be more than mental gymnastics; that it must draw its content from commercial and industrial life; that apart from explaining social practice and answering social needs, it has no place or purpose in the course of study. Recent studies on transfer have rendered no little aid in ridding the elementary-school course in arithmetic of its arbitrary character, and in eliminating from it much of the unnecessary work in tables of uncommon weights and measures, complex fractions, calculations with fractions whose denominators are absurdly large, impossible reductions ascending and descending in denominate numbers, extreme rationalisations in early stages of number work—in short, the socially useless matter that was retained because of false psychology and deference for tradition. All these must go. The course of study must be simplified by rigorously excluding all those topics that are not possessed of social use. It must be enriched, first, by introducing the facts and practices of modern life, and, second, by a vitalized type of teaching, which will develop habits of accuracy, of concentration, and of planned and purposeful thinking. The disciplinary values should come as by-products—inevitable, to be sure, but by-products none the less—of teaching, which concerns itself with subject-matter that is genuinely and vitally related to the business of living.

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[11] The reader will find a worthwhile discussion of the disciplinary values in the *Report* of the National Committee on the Reorganization of Mathematics in Secondary Education (The Mathematical Association of America, 1923), pp. 8-9.

Pleasure values of arithmetic. The economy of life demands a mastery of the world on the quantitative side. In our very early days we learn to gauge effort in terms of physical force to be overcome. The infant in his bed soon learns just what exertion to make to reach the rungs of his crib. Primitive man, aiming his crude bow, unconsciously slackens or tightens the string depending upon the distance and the direction in which his prey is going. In the simplest activities—in tearing, pulling, lifting, reaching, clutching—we unconsciously strike a balance between the energy we must expend and the physical resistance we must overcome. It would seem as if nature, realising that human wants are legion and human energy decidedly limited, set up these mechanisms within us whereby we unconsciously balance energy and effort, lest we find ourselves bankrupt in vitality, with life's needs still pressing upon us. While we really never know nature's real intent, we cannot, of course, resist the temptation to speculate about her purpose, and to rationalise her behavior by man-made logic.

The origin of number has often been traced to an innate rhythmic sense. This is not the place to discuss whether we bring an awareness of rhythm to life, or whether the rhythmic experiences of life instil this rhythmic sense in us. We have ample evidence for the assertion that very young children exhibit unmistakable preference for rhythmic repetitions. Pupils in the introductory grades enjoy the rhythmic repetitions in counting:

5, 10, 15, 20, 25, 30, 35; 2, 4, 6, 8, 10; 22, 24, 26, 28.

They learn much about number when they become aware of the rhythmic patterns in their skipping and hopping games. Few adults, walking beside a fence, can resist the temptation of touching every alternate, every third, or every fourth picket; or, walking on a flagged sidewalk, of stepping only on alternate cracks. Who can maintain an even backward and forward movement while rocking in a porch chair? Before long we accent every forward movement or every alternate backward movement. In the sleepless hours of the night the tick-tock of the clock soon assumes a rhythm of accented and unaccented ticking that is as metrical as simple poetry. What are these satisfying rhythmical experiences but aspects of number?

A puzzle is usually a challenge that we refuse to ignore. Our minds wander back to the unsolved puzzle when we may be most eager to focus thought on a real problem. What valuable hours have we spent resolving the predicament of the man who had to ferry the goat, the cabbage, and the wolf across the stream, one at a time, but could not leave more than one of these on the other bank lest the goat eat the cabbage or the wolf devour the goat. Drills in the mechanics of arithmetic can be made interesting by organizing them as number games. In presenting the short cut for multiplying by 12½, 25, and 125, a teacher did not announce that the day's lesson would teach a new way of multiplying. She was convinced that most children feel one way is quite adequate. Instead the teacher asked the class to try the multiplications indicated on the blackboard to discover whether any pupil can obtain the answer as rapidly as she. All began at signal.

1842 × 12½ 1842 × 125

The teacher wrote her answers and put her pencil down with enough noise to attract the attention of the class. The children, surprised at her speed, continued to work. At the completion of the task, answers were compared. The teacher's were correct, of course. Then hands were raised. "How did you get the answers so fast?" The teacher then asked, "Do you think I multiplied by ½, then by 12, and then added?" She followed their negative answer by, "Then how did I obtain the answers to both examples in twenty seconds?" The children knew, "You have a trick." Of course they were eager to learn the trick for they wanted to use it on their friends. Arithmetic situations, properly presented, invite solution because they are puzzle experiences to children.

Who has not experienced the joy of achievement? An adequate psychology of success has yet to be written. The challenge of an arithmetical situation holds out hope of successful solution. The joy of finding one's answer correct and one's decisions verified are genuine motivating factors because they appeal to much that has become second nature in us.

To some children numerical processes and symbols serve to stir vivid and quaint imagery. Pupils tell us that 4 represents a fat man and 7 a tall lady; that 9 calls up the picture of a lazy boy; that 6 tries to "break into 24 but 4 stops it and it finally breaks into 42 because 7 that was supposed to watch it was too weak to stop it." In many tests pupils seemed to prefer odd numbers in units' place and even numbers in tens' place. The numbers three, six, seven, and nine play important rôles in mythology and early religious ritualism.

The unconscious desire to acquire a mastery of the quantitative phases of life, the rhythmic sense, the lure of the puzzle, the promise of successful achievement, the image-stimulating power of numbers—all coöperate to give the child a pleasure in mastering number facts and number processes.

Teachers must aim to utilise this natural and conditioned preference for number, and to endeavor to relate their teaching of arithmetic to the child's fund of interests. To neglect these is to reduce number study to a mechanical and arbitrary series of symbols and processes.

Cultural value of arithmetic. Society frequently makes arbitrary demands of education; adherence to them can be explained on traditional grounds only. We expect people to know, even though they may have no occasion to employ such expressions as *the quotient figure, changing rates, proportional relation, reciprocal, perimeter,* and *as basic as the square on the hypotenuse.* To a limited extent we must recognise and satisfy the cultural demand, but we must hold it in check and not permit it to assume a controlling place in teaching arithmetic or in shaping the course of study.

It is not always simple to dissociate the cultural from the social value. The latter has its legitimate claim. The former refers, in our discussion, to more or less arbitrary standards set up by society. In a world in which science and the scientific method are assuming ever-increasing significance, mastery of number concepts and of the techniques of elementary mathematics is a primary function of education. Our quarrel is with subject-matter that no longer functions in life.

Preparatory value of arithmetic. A final value that is often urged for arithmetic is its ability to prepare the child for later studies. The child who knows the fundamentals of arithmetic can solve some of the quantitative problems that arise in physics and can grasp with greater ease the basic operations of algebra. The teacher of bookkeeping and accountancy finds his task complicated by pupils' lack of mastery of the fundamental facts and skills of arithmetic. Teachers of trade subjects complain bitterly that this same lack retards their class work. Before teaching the technical applications of arithmetic to various crafts these teachers must give drills designed to help the adolescent school graduates understand the obvious difference between ⅛ of 12½ and 12½ ÷ ⅛, or to lead them to understand that .1 is more than .098. In these days of continuation schools, trade schools, and commercial high schools; the preparatory value of arithmetic is vital for it affects an ever-increasing number of boys and girls who are being prepared for a definite calling. We do not urge that the arithmetic of the elementary school be vocationalized by the introduction of a variety of technical applications, but rather that fundamental facts and processes be so thoroughly mastered that the graduates will bring an adequate basis of skills to their teachers of algebra, of bookkeeping, or of the trades.

Summary and conclusions. Our study has sought to justify the teaching of arithmetic on a fivefold basis: the practical, the disciplinary, the pleasure, the conventional, and the preparatory. We have presented here no coördinate set of values. There is a relativity of worth that must be clearly perceived and constantly kept in view as the guiding principle in teaching and selecting the subject-matter of arithmetic. We are rapidly making for a new arithmetic. The present-day interpretation of the doctrine of the transfer of abilities, the view of education as a socializing function, the demands of industry, the more sympathetic comprehension of child life—all these are coöperating to humanize the subject, and to teach that the practical value of arithmetic is primary. To it, all other values must bend. A course of study in arithmetic, selected and organized with the utilitarian aim in view, may be so taught that all other values are attained to their fullest measure.

PROBLEMS FOR STUDY AND DISCUSSION

1. Select a course of study in arithmetic with which you are familiar. To what extent has each of the objectives discussed in this chapter been a controlling factor in the formulation of this course of study?

2. Prepare a test on Chapter I composed of at least, (*a*) 20 true-false statements, (*b*) 10 completion questions, (*c*) 5 multiple-choice questions. Before beginning this task review the chapter and list what you consider its basic facts. Would you regard the score which a fellow student might make in such a test as a reliable measure of the degree of his mastery of this chapter? Why?

3. Consult the bibliography at the end of the book and select

three references for further study of the content of this chapter. What will guide your selection?

4. In a sense, arithmetic should be taught for the same reasons that govern the teaching of the social or literary studies: to help the child live a more abundant life. What is meant by a *more abundant life*? Give illustrations. Do you agree with the statement? Justify your position in a carefully planned exposition.

5. Did you enjoy learning arithmetic and elementary mathematics? Why? (Think of the objectives your teachers were probably trying to attain, the content of the courses, and the methods of teaching and testing.) What interests, if any, did you bring to the work? Were these interests satisfied or utilized? Be specific.

6. An American law school of high repute urges college students who expect to study law to concentrate in mathematics. How could the faculty probably justify this advice? What facts in this chapter would lead you to challenge this advice? Formulate your own position, first to yourself, and then in four or five carefully written sentences.

7. The objectives of any school study, however carefully formulated, are never mutually exclusive but shade into one another. Illustrate this by reference to this chapter.

8. List three major objectives that are common to the teaching of reading and arithmetic.

PART II

THE COURSE OF STUDY

CHAPTER II

THE CONTENT OF THE COURSE OF STUDY

The course of study a determining factor in efficient teaching. The course of study in arithmetic is one of the basic factors in determining the efficiency of the school course in this subject. If the course of study errs in the selection of subject-matter, teachers and pupils spend valuable time on content of no social worth; if the course of study is organized with no regard to sequence and gradation, the children's minds are confused by the rapid succession of difficulties; if the course of study does not indicate a proper allotment of time to mechanical and to thought exercises in arithmetic, children are overdrilled in the fundamental operations, and are rendered helpless in arithmetical situations that deviate from the type form; if the course of study sets a fixed limit for all children in a given grade, some are discouraged by being required to attain a standard that is unreasonably high in relation to their abilities, while others, more gifted, are marking time. In the final analysis the course of study determines the possibilities of introducing effective modes of instruction, and of developing in our school graduates such arithmetical ability as will meet life's needs.

The course of study must be definite in its requirements. Much confusion arises from general statements and indefinite requirements in a course of study in arithmetic. The following assignment is an illustration of an inadequate course-of-study prescription;

Oral. Daily drills to develop power, accuracy, and alertness. Oral work to review, introduce, and supplement written work, involving both abstract numbers and concrete problems. Terms—base, rate, and percentage.

Written. Integers—Daily practice in the fundamental operations, with attention to accuracy and rapidity.

Fractions. Addition, subtraction, multiplication, and division of fractions and mixed numbers. Principles: multiplying numerator, multiplies fraction; multiplying denominator, divides fraction; dividing numerator, divides fraction; dividing denominator, multiplies fraction.

The weaknesses of such prescriptions become evident as one tries to carry them out. Each succeeding assignment expresses the same futile hope that "power, accuracy, and alertness" will be developed, and yet no teacher, in any one grade, knows what degree of "power, accuracy, and alertness" her pupils are expected to attain. At no place in the course of study are these terms defined. If these objectives are real goals, then teachers must have approximately similar conceptions of these qualities, and must know what degree of skill her pupils may reasonably be expected to achieve in each of them. We must avoid such statements as, "develop proficiency in multiplying and dividing by decimals, inculcate habits of systematic arrangement of work, lead children to plan work before beginning any operation," and the like. They are all so general as to be meaningless. To be valuable a prescription must be set forth with quantitative norms representing actual average achievements of children in a given grade, or of children of a given age.

Let us illustrate the type of definite assignment that may reasonably be expected in a course of study.

Sample of Task to Be Accomplished with x Per Cent of Accuracy	Number of Examples	Time in Minutes
1. To add 4 8 7 5 2 3 Sum less than 20	16	2
2. To add 7 6 6 5 0 5 etc. Twelve addends; single column	5	4

Sample of Task to Be Accomplished with x Per Cent of Accuracy	Number of Examples	Time in Minutes
3. Ability to divide by 7, 8, 9; dividend 3 or 4 places; first place in dividend smaller than divisor; no remainder. Example $8\overline{)3840}$	10	10
4. To add 3 fractions, dissimilar denominators, but common denominator easily obtained by inspection. $\frac{3}{4}$ $\frac{5}{8}$ $\frac{1}{2}$	5	10
5. Two-step problems of the addition-subtraction type, sum less than 25, paper and pencil, for example: Bought a 12-cent loaf of bread and a 9-cent can of peas. How much change do I receive from 25 cents?	10	15

To be sure, there is danger in any system of supervision that sets up standards excessively specific: teachers may curb the originality of their pupils or their desire to exceed the specific demand. But the advantages of goals definitely and quantitatively defined seem to outweigh the dangers, especially when teachers are made to understand that these standards represent average attainment.

How can definiteness of course of study be attained? Let us now consider a few simple devices, which introduce into a course of study a degree of definiteness that will help to insure concentrated effort toward clearly conceived ends.

1. Terms must be defined. As we have seen, courses of study must not ask teachers to develop "alertness, power, good arithmetical habits, proper algorisms, modes of thought in arithmetic, modes of attack." These expressions have no fixed connotation and vary with the past experience and the professional training of each teacher. To some teachers *alertness* may mean speedy mental reactions; to others it may mean sagacity.

2. Quantitative norms with definite illustrations must be given as the approximate average performance to be expected in successive grades. Assuming that the terms taken from courses of study are actually defined and illustrated, it is evident that the degree of proficiency varies with each succeeding grade. Accuracy in addition in the fourth year is far below that expected of a fifth-year class. How is a teacher to know what degree of neatness, accuracy, speed, and ability to meet new arithmetical situations is expected of her pupils at the end of a school term? How is a teacher to know what measure of these techniques she may expect at the beginning of a grade? Specimens of children's papers ought to be submitted in the course of study so that teachers will know the degree of neatness expected each succeeding year. Experimental work in establishing quantitative norms in arithmetic has gone far enough to warrant progressive school authorities in setting up the best of these as tentative standards for their school systems.[1]

3. Throughout the grades types of problems should be identified by illustrations, not by classification. To assign, "Teach Case II in Percentage," does not delimit the scope of the field. Strictly speaking, each of the following examples belongs to Case II:

a. Balls listed at $4.80 a dozen are bought at a discount of $1.20 by a dealer. What is the rate of discount allowed to dealers?

b. Tennis balls listed at $4.80 a dozen in the catalog cost the storekeeper only $3.60. What is the rate of discount allowed the storekeeper?

c. A dealer paid $3.60 for a dozen tennis balls. He was allowed a discount of $1.20. What rate of discount are dealers allowed?

Which of these types of Case II should the children be taught? The first—the simplest only? Will that meet the demand of the course of study? Must every reasonable variation of Case II be taught to all pupils or only to the more gifted?

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[1] For detailed study of quantitative norms and scientific evaluations, see Chapter XIV.

These questions should be answered by the makers of the course of study.

4. Definiteness may be attained by prescribing uniform procedures and algorisms. Some prefer the form of $\overline{)\quad\quad}$ for all divisions, others advocate its use only in long division but retain the old form $\underline{)\quad\quad}$ for divisors under 13. One teacher encourages the writing of "carry figures" in addition, another opposes the practice vigorously. Of the various sanctioned arrangements of work in adding and subtracting fractions with dissimilar denominators, one should be prescribed throughout the grades so that children will not be required to unlearn one algorism and learn another with each change of teacher. Some courses of study make references to textbooks that are frequently found on teachers' desks in order to illustrate each prescribed form. The following are examples of this device taken from better courses:

Reduction of fractions to decimals: use no fractions with terms larger than those found in Th's ————, pages x-y.

Addition of fractions: follow the gradation and arrangement suggested in H. and P's ————, sections *a* and *b*, pages x-y.

5. When a subject is to be taught through a number of grades, there must be a limited allotment for each grade. Thus one finds, *Business Forms, Tables of Weights and Measures,* and *Abbreviations Commonly Used in Business,* assigned in a number of grades. If business forms are to be taught in 5A, 5B, and 6A, it is obvious that each grade teacher must know the limits of her responsibility. Under the general assignment, each teacher teaches all the business forms so that the pupils may be ready for any business form that may be given in the examination set by her supervisors. With this undue increase in the scope of work in any grade, time for habituation is insufficient; the forms are learned imperfectly and must be retaught in the succeeding grades, but with no better outcome.

The Content of Courses in Arithmetic

What determines the content of the course? Two principles are recognized as guides in determining the content of the course of study in arithmetic. The first is the child's interests. That certain interests of children are satisfied by number experiences we have demonstrated. Young people frequently do not know their interests. As they get into new experiences, new interests manifest themselves. The child is entitled to live his present life, but, in a deeper sense, his life is only a bridge to his real life—the life of adulthood. Hence a second principle is introduced to help in selecting suitable subject-matter—the demands of society. The subject-matter must transcend the immediate interests of the child; social need must be met regardless of the temporary interests and attitudes of childhood. If only children's interests prevailed, courses of study would be filled with puzzles and number games almost to the exclusion of the significant phases of the subject. In teaching arithmetic we must show the child the social need of the topics under discussion. If the child is interested, so much the better. If not, he must learn them anyhow because life demands them. This may not be pleasant learning but it is motivated and rationalized learning.

Frequency of occurrence as a measure of social need. We are ready to grant the priority of social need as a measure of importance of curricular material. But how may we determine what is socially necessary? An objective measure of social need, according to many, is frequency of occurrence in the daily regimen of life. The argument runs, "That which occurs most often is most important."

Studies have been made of the frequency with which certain number facts appear in everyday life. What if a count were to reveal that $2+3$ is ten times as frequent as $8+9$? Ninety-six per cent of all the fractions which occurred in one study consisted of ½, ¼, ¾, ⅓, ⅔, ⅛, ⅜, ⅕, ⅖, ⅘. In another study, 93.9 per cent of all the fractions consisted of ½, ⅓, ⅔, ¼, ¾, ⅕, ⅛.[2] What are we to conclude from these facts derived objectively and with much travail? Are we to neglect 8+9 in our teaching, or supply for 2+3 ten times as much drill material as for 8+9? Is the correct answer to 8+9 therefore only one-tenth as important as the correct answer to 2+3? Are we to neglect the fraction $\frac{3}{5}$ because it is in neither list? Fractions with denominators of 12, 16, 18, 32, and 64 do occur in non-technical content. Shall they be excluded from our teaching?

It is not altogether unwise to argue that what occurs most frequently need not be emphasized in teaching. Life will teach the most common facts by natural repetition; the relatively infrequent but necessary facts will not be learned from everyday living. Hence some teachers are driven to conclude that they should stress, in class, the less frequent and the more difficult. This gives us, again, a conclusion that is not free from sophistry.

Urgency, not frequency, is urged as a measure of social significance. How often does one have to swim? Yet swimming skill is not less important than other skills because the need for it is infrequent. The very significance of an emergency lies not only in its unexpectedness, but in this infrequency of need.

We are not very near to a solution of the question, "What constitutes social utility?" To be sure, our objective studies have yielded much data, but qualitatively they are not adequate to serve as a basis for a final judgment. Many of these studies, though sincere in objectives and scientific in techniques, have not risen above the level of professional busy work.

Flexibility of courses of study. A sound curriculum is constructed so that it can be adjusted to meet varying needs and capacities. To teach all pupils the same subject-matter is

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[2] H. E. Mitchell, "Some Social Demands on the Course of Study in Arithmetic," in the *Seventeenth Yearbook* of the National Society for the Study of Education, Part I (Public School Publishing Co., 1918), pp. 7-17. H. E. Benz, "The Arithmetic of Department Store Transactions," Master's thesis, University of Iowa. Quoted in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education, p. 44.

neither possible nor desirable. Children of limited intelligence, who will leave school as soon as the compulsory-educational law permits, may well be spared the discouragement that would attend their attempt at learning multiplication by 20 305, division by mixed numbers, and decimals beyond three places. Nor is it necessary that they know how to find areas of triangles, or how to find the base when rate and percentage are given.

It is well for a course of study in arithmetic to indicate three levels of achievement—one for the average, another for the slow and retarded, and still another for the superior child. A norm, quantitative in form and objectively derived, should be given for each of these three types of pupils. With the growing tendency toward homogeneous grouping such a differentiated course of study is absolutely essential. The present practice is to set down for each grade the assignment for "the average class of normal-progress pupils." The following is an example of such a grade prescription, taken from a carefully devised course of study:

5B (Fifth Year, Second Half)

Course of Study

Oral

Reading numbers
Common fractions
Measurement
Problems
Review

Written

Writing numbers
Common fractions
Decimals
Measurement
Graphs
Problems
Review

Syllabus

Oral

Reading Numbers
Four-place decimals

Common Fractions
- Changing a mixed number to an improper fraction, the integral part not to exceed 9 and the denominator of the fractional part not to exceed 10
- Changing an improper fraction to a whole or mixed number, the numerator not to exceed 99 and the denominator not to exceed 10
- Multiplication of fractions * by multiples of the denominators, multiples not exceeding 100; by non-multiples not exceeding 12. Multiplication of a fraction * by a fraction *; finding a fractional part * of a fraction *
- Finding what fractional part one number is of another, neither to exceed 36

Measurement
- Table of United States money
- Use of inch, half-inch, quarter-inch in measuring lines and small common objects

Written

Writing Numbers
- Four-place decimals

Common Fractions
- Addition and subtraction of fractions and mixed numbers, the number of addends not to exceed 6
- Changing a mixed number to an improper fraction
- Multiplication of fractions and mixed numbers
- Finding what fractional part one number is of another

Decimals
- Addition and subtraction, limited to four decimal places, the number of addends not to exceed 6
- Multiplication: decimal in multiplicand, in multiplier, in both
- Fraction in product limited to four decimal places
- Effect on decimal of multiplying by 10, 100, 1000

Measurement
- Interpretation of simple scale drawings; determining distances represented on maps used in the geography of the grade by means of the scales to be found on these maps

Graphs
- Line graphs of pupil progress constructed and interpreted by the pupil

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* Denominators not to exceed 12.

Oral and Written

Problems
- Problems should arise from projects or activities. The activities and problems should provide for the introduction and the significant use of all the formal facts and processes of the grade, the problems being kept within the limits of the formal work. Include problems without numbers. One and two operations.

Review
- Frequent drills on the fundamental combinations and processes, and other work of previous grades to develop accuracy, facility, and speed
- Remedial practice for the correction of weaknesses in computation and problem-solving as revealed by standard diagnostic and other tests

The slow child may well be limited to decimals of two places, but the gifted child learns not only to read and write decimals of four places, but acquires so thorough an understanding of the principles of our whole system of notation that decimal fractions appear merely as an extension of a notational system of whole numbers. In dividing by a decimal the work must also be suited to varying capacities. The slow pupil may learn the process through dint of practice, and may even be limited to dividing dollars and cents by a whole number. The normal pupil should meet the demands of the course of study. The gifted child should learn the process with that degree of understanding which would enable him to divide 87 546 by 600 by first dividing both dividend and divisor by 100, and then proceeding to divide by 6. The differentiated assignment, therefore, is not solely quantitative; it is qualitative as well, for it seeks to give an insight into numerical relationships that not all children can comprehend. An illustration of this type of flexible course of study is offered in table on page 33.

The undifferentiated course of study does serious injustice to the gifted child as well as to the slow one. The former is limited to the assignment for the normal pupil. He soon grasps this material and is then forced to mark time while his slower classmates struggle for mastery of a process which came to

SAMPLE OF FLEXIBLE COURSE OF STUDY WITH DIFFERENTIATED ASSIGNMENTS

(Number of Examples, Time Limit, and Character of Examples to Be Derived by Experimental Data)

Sample of Task to be Accomplished with *x* Per Cent Accuracy	Norm for the Average Child		Norm for the Slow Child		Norm for the Gifted Child	
	Number of Examples	Time in Minutes	Number of Examples	Time in Minutes	Number of Examples	Time in Minutes
1. Ability to read and write decimals with and without whole numbers	4 places		2 places		4 to 6 places with thorough understanding	
2. Ability to add common fractions, dissimilar denominators, common denominator found by inspection	6 of 3 addends each	8	6 of 3 addends each	10	6 of 3 or 4 addends each	6
3. Ability to subtract, 2 borrowings, decimals, for example, 58.42 39.24	3	1½	3 Habituation with no rationalization	2	3 Habituation with complete rationalization	1
4. Finding what fractional part one number is of another, neither number to be over 36. A. Had 24, lost 8; what part was lost? B. Had 24, had left 16; part lost? C. Had left 16, lost 8, what part lost? Examples presented in written or printed form. Answer to be given by inspection.	5 3 of type A; 2 of type B	2	4 of type A only	2	6 2 of each of types A, B, C	2

him without appreciable effort. Aside from the loss of his time the gifted child finds no mental stimulation in a subject that should be a challenge to him. The lot of the slow child is no happier. He is expected to attain a level of proficiency which is beyond him; he is harassed and beset with discouragement; little by little he becomes convinced of his inferiority. A properly organized course of study does full justice to the children at both ends of the scale of ability. By indicating the minimum requirement it insures those pupils who are destined to limited schooling a mastery of those phases of arithmetic most useful in meeting the demands of everyday life.

Cautions in formulating minimum courses of study. The practice of formulating minimum expectations for each grade is not without its dangers. Under inefficient supervision the minimum too often becomes the maximum requirement, and the class as a whole is reduced to the level of the mediocre or, worse, of the backward. Supervisors must exercise great care to insure the full operation of a differentiated course. Only with the knowledge of the supervisor should children be limited to the minimum requirements.

A second danger in the differentiated curriculum is that the maximum course may include much that has no social value. The same standard, social utility, must control in formulating both the minimum and the maximum course, the former containing what is relatively more useful, the latter, what is relatively less useful, but nevertheless unmistakably useful. At no time must the maximum course degenerate into a catch-all of puzzles and obsolescent computations that are justified because they afford mental gymnastics to young minds of promise. The maximum course must give the superior child a deeper insight into number relations, a higher degree of proficiency in fundamental operations, a veritable hierarchy of good working habits, and an adequate understanding of business customs and practices.

The content of the minimum course of study. Selecting on the basis of relative social need, we find an irreducible minimum without which a child can neither understand the most common quantitative aspects of his environment, nor intelligently approach the arithmetical tasks that make up the day's routine in the usual activities of life. The absolute minimum must contain the following:

1. Fundamental operations (addition, subtraction, multiplication, division) with whole numbers. Multipliers and divi-

sors not larger than 4 places. Terminal and middle zeros included.

2. Fundamental operations with common fractions. The denominators limited to 2, 3, 4, 5, 6, 8, 9, 10, 12. Cancellation.
3. Fundamental operations with decimals. Limited to three places. Emphasis on United States money.
4. Problems:
   Fractions; to find fractional part of whole number, fraction, mixed number.
   Decimals; to find decimal part of whole number, decimal, mixed number.
   Percentage; to find per cent of any reasonable quantity.
   Finding cost of a quantity, given the rate per 3, 5, dozen, 100, 1000.
   Problems requiring keeping of a budget.
5. Percentage:
   Direct case (First Case) of any of the common business applications of percentage, namely, profit and loss, commission and brokerage, interest (years and months), commercial discount.
6. Business forms:
   Bills, receipts, receipted bills, checks, stubs in check book, deposit slip, cash-book account.
7. Tables of weights and measures:
   About six of the commonly used tables. Selection of the tables to be determined by the needs of the locality. Simple computations with these units limited to two steps in changing from larger to smaller units, and vice versa.
8. Mensuration:
   Finding areas of rectangular figures and volumes of cubes, rectangular forms like boxes and rooms.
9. Special applications of fundamental processes to industries and business of the local community.
10. Simple graphs and statistics:
    Bar and line graphs. Averages. Reading of tabulated material.

We have here no course of study, but an outline of those topics that are usually considered the absolute minimum in meeting the most pressing arithmetical needs. Those who use this tabulation of topics will find it necessary to make suitable allocation of materials to the different grades; to correlate the processes of any one grade, so that the class learning to multiply whole numbers and fractions will also learn how to apply this knowledge in making out bills; to define with great clearness how intensively each of these topics is to be taught; to set up a concrete or a quantitative measure of the approximate degree of proficiency to be attained at the end of any one year. This outline, it must be reëmphasized, is not a course of study; it is only a guide in formulating the minimum arithmetical requirements.

Provision for individual differences. We have urged differentiation of training for children of different levels of ability. A study made by the Denver schools leads the authorities to conclude, "The assumption is true that children of the same mental age in the normal and borderline groups do practically the same grade of work until the mental age of eleven is reached."[3] We accept the findings but we must not forget that the test was limited to mechanics of arithmetic and to the operations with whole numbers. No evidence is presented to show the reactions to common and decimal fractions. It is probably uneconomical to teach the borderline child to add beyond two columns of 5 addends, or to subtract four place numbers with all the zero difficulties.

Provision for individual differences may be made in various ways:

1. Use of minimal, average, and enriched day-by-day assignments.
2. Teaching from three different courses of study designed for children who are slow, average, and gifted, respectively.
3. Teaching the same basic course but with different rates of progress for the different groups. The gifted group is expected to complete the course in far less time than the average group, and the slow group in more time than is required by average children.
4. Varying degrees of efficiency may be required. The passing grade changes with the ability of the group.
5. Use of large units of work in which each pupil carries out that part most suited to his abilities and interests. In the discussion of the project appropriate illustrations are given.

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[3] West, Greene, and Brownell, "The Arithmetic Curriculum," in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education (Public School Publishing Co., 1930), p. 121.

6. Individualization of work through self-study as in the Dalton plan. A later chapter dealing with recitation patterns discusses more fully this mode of teaching.

The general course of study. *How complete a study of number shall be made.* Teachers of arithmetic are divided into two distinct camps concerning the completeness of the course. One is uncompromising in its insistence upon the elimination of all but the most useful facts and processes. The second group would give a reasonably complete study of number and number processes. This latter camp argues that no understanding of the simplest relationships is possible without a varied contact with number; that we have no means of foretelling the pupils' arithmetical needs in their future vocational or personal relations; that an adequate sense of accuracy is impossible without an experience with number that transcends the mere utilitarian phases. To these teachers arithmetic is not solely a tool subject. It is not and must not be made an end in itself, for arithmetical skills and numerical thinking are part and parcel of thinking in almost every other experience. To limit arithmetic, as the utilitarians insist, would destroy the true educational significance of a time-honored school subject.

The limits of a reasonably complete study of number are not easily defined. The following outline probably combines the best judgments of those who have given earnest thought to the problem, and is here presented, year by year, with indications of the extent to which personal opinion varies.

FIRST YEAR: Incidental number work through games, music, manual training. Introduction of ideas of relative magnitude: little, less, more, higher, lower, longer, shorter, much more, much less, etc. Simple units, cent, foot.

Degrees of variations: From no number work to formal drills in addition, subtraction, and simple multiplication.

SECOND YEAR: Number work through correlation. Addition and subtraction emphasized. Multiplication tables begun. Common Units: foot, yard, cent, nickel, dime, pound. Also measurement with defined units that are not standardized, such as cupful, hand, pace, etc.

Variations: From no number work to drill on simple division.

THIRD YEAR: Addition and subtraction mastered. Much time to multiplication and division tables. Further work with defined and standardized units of measure.

Variations: From mere introduction of arithmetic to considerable drill in fractions that can be added and subtracted by inspection.

FOURTH YEAR: Multiplication and division mastered. Informal work in fractions by inspection. Denominate numbers continued.

Variations: From mere introduction of formal computations and incidental work in fractions to formal work in fractions.

FIFTH YEAR: Fractions mastered. Decimals introduced; denominate numbers; perimeters and areas of rectangles.

Variations: Too wide to be formulated.

SIXTH YEAR: Decimals mastered. Percentage and profit and loss. Denominate numbers. Business forms.

Variations: So great as to permit of no systematization.

Aim of the First Six Years: To develop skill in the fundamental operations as applied to whole numbers, common and decimal fractions, and to teach the necessary number facts.

SEVENTH AND EIGHTH YEARS: Percentage and every variety of business application. Business forms and an understanding of business practices, especially banking and the principles governing sound investments.[4] Mensuration applied with great variations in extensiveness of treatment, greatly reduced in best courses of study. Simple ratio and proportion. Longitude and time. Metric system. Square root by inspection. Simple statistics; keeping of accounts and budgets. Drawing to scale. Reading simple floor plans. Reading of meters. Post-office arithmetic—money orders, parcel post costs, etc.

Aim of Last Two Years: To develop power to understand arithmetical solutions and manipulative quantities in problems varying from the type.

Summary. This composite course of study, whose constituent courses represent the best efforts of supervisors in representative communities in the United States, reveals clearly that personal opinion still controls in the selection of curricular material. As was shown, inquiry into social needs has not resulted in a clearly defined standard in terms of which subject-matter may be selected and rejected. The sequence of topics and the gradation of skills need no longer be left to personal judgment. Experimental educational psychologists have combed the field, and have identified the specific skills that constitute general ability in arithmetic. These matters will receive adequate treatment in later portions of the book, which present methods of teaching each of the important processes in arithmetic.

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[4] Such arithmetic is studied with a minimum of computation. Thus, in banking and interest, the following are discussed: What is a bank; what does the bank do with the many daily deposits; why banks pay interest; the legal rate; the safe rate on investments; the differences between saving and checking accounts; how to compute interest; how to write a check; how and why the stub is filled out; when a check may be cashed; how to endorse a check; how to make deposits and fill out the deposit slip; what to do when a check is lost; how to stop payment; certified checks; the monthly statement.

What to omit from the traditional course of study. Analysis of existing courses shows clearly that most of them have undergone serious simplification in the last decade by elimination of much of the content that seems no longer to play a part in the business life of our day. Progress in the teaching of arithmetic, generally speaking, is no longer retarded by the presence of obsolete subject-matter. Two factors are responsible for this change: our restricted faith in the doctrine of transference, and the inability of trained investigators to discover what, by common agreement, must be included to satisfy social needs. There is, however, reasonably common agreement that the following topics be excluded from the general elementary course in arithmetic:

1. In Fractions
    - *a.* All fractions whose denominators are larger than sixteen, or, if smaller, are unusual in actual occurrence, for example, fifteen, thirteen, seven, and eleven.
    - *b.* Greatest common divisor and least common multiple or denominator that cannot be found by inspection.
    - *c.* Ratio beyond the mere interpretation and the expression in fractional form.
    - *d.* Complex fractions.
    - *e.* Compound fractions should not be eliminated. The form $\frac{2}{3}$ of $\frac{3}{4}$ is useful.
2. Denominate numbers
    - *a.* All uncommon tables of weights and measures, for example,

Troy weight, Apothecaries' weight, Surveyors' measures, etc.

b. Units of measure, obsolete, imaginary, or not used in the child's environment, for example, long ton, long hundredweight, small measure (half bushel), hand, rood, furlong, etc.

c. Reduction ascending and descending, division and multiplication of denominate numbers involving more than three units, for example, "Reduce 1 mile, 2 rods, 3 yards, 2 feet, 7 inches to inches." "Reduce 5,246,489 inches to high denominations." "How much will 72 barrels hold, if the capacity of 1 barrel is 15 gallons, 3 quarts, 1 pint, and 2 gills?"

3. Percentage

a. Unusual and obsolete business applications, such as partial payments; true discount; interest in cases where the rate is not used in actual business, or in cases where principal must be found; life and fire insurances computed by rate per cent on the dollar, thus: "Find premium on a fire- or life-insurance policy, the rate being ⅜ per cent; annual interest."

4. Problems

a. Indirect Cases, that is, Cases II and III of percentage. Many teachers still favor the inclusion of these indirect cases, but would excuse backward children from learning these.

b. Partnership. Corporate organizations more common, hence partnership with time may be omitted.

c. Compound proportion: Examples of this type: "If 54 men working 20 days build a wall 750 × 15 × 2 feet, how many men are required to build a wall measuring 250 × 12 × 2 feet in 30 days?" Some authorities would omit all proportion.

d. Problems that form no part of industrial and commercial practice but are given as mathematical puzzles to challenge thought and ingenuity.

5. Mensuration

a. Areas and volumes of uncommon figures, for example, area of trapezoid, surface of cone, sphere; volume of cone, pyramid, sphere, frustum of cone. Such topics should be included in a secondary-school course in geometry or vocationalized course in arithmetic.

6. Vocationalized arithmetic

a. The technical procedures and computations of specific and specialized industries should not be made the subject for study by all children. Only those phases of arithmetic which contribute to the understanding of the common experiences of life should be incorporated in a general course. Examples in plastering, carpeting, paperhanging, gilding, etc., are subject to specialized trade practices and should be omitted.

7. Miscellaneous subjects
   - *a.* Square root of number too large to be solved by inspection or factoring.
   - *b.* Cube root.
   - *c.* Longitude and time beyond simple case of computing approximate difference, allowing 15 degrees to an hour. Kept simple, this subject is both useful and interesting.
   - *d.* Metric system. The generally advocated practice is to teach the metric system in science where the pupils apply the new units immediately in the laboratory exercises or in their science problems.
   - *e.* Foreign Money. Children should learn the names and the approximate value of the unit of the most common monetary systems, for example, £, franc, lire, mark.[5]

Courses in arithmetic have grown in content, gradually but persistently, until the channel of subject-matter has increased in width but has suffered a proportional diminution in depth. In formulating the table of what can be profitably deleted from traditional courses of study, the author has suggested only those subjects and processes which are generally condemned by recognized students in the field. Too much has been attempted in arithmetic, and the result has been an inevitable superficiality of understanding and serious lack of skill in fundamental operations. Simplifying the course of study, by eliminating material that is of comparatively little social use, will narrow the channel of knowledge and afford the teacher adequate time to insure thoroughness of comprehension and skill of manipulation.

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[5] A full treatment of the reduction of the course of study is found in "Curricular Problems and Their Scientific Solution," in the *Third Yearbook,* of the Department of Superintendence of the National Education Association (Washington, D. C., 1925), pp. 35–110. Also in G. M. Wilson, *Connersville Course of Study in Mathematics for the Elementary Grades* (Warwick and York, 1922), pp. 13-14.

The unified junior-high-school course in elementary mathematics. Soon after the beginning of this century elementary schools consisting of Grades 1–8 made it a common practice to introduce algebra in the seventh or eighth grades. In 1915 about 35 per cent of the city school systems taught algebra in the last two grades as an independent subject. With the development of the junior high school the need of a new unit of mathematics became apparent. Since the arithmetic taught in the traditional eight-year organization is not satisfactorily completed by the end of the sixth year, junior-high-school mathematics usually continues the study of arithmetic. The weakness of superimposing a year of unrelated algebra upon eight years of arithmetic became apparent, and gave rise to a unified course in elementary mathematics for the junior high school. This consists of interrelated work in arithmetic, algebra, intuitive or experimental geometry,[6] and simple trigonometry. The Report of the National Committee on Mathematical Requirements[7] suggests five plans for the reorganization of the work in elementary mathematics. All plans include applied arithmetic and intuitive geometry for the seventh year, algebra for the eighth year, trigonometry for the ninth year, while four advocate demonstrative geometry for the ninth year.

The vital problem is to decide whether we shall evolve a unified and integrated course in elementary mathematics for these grades, or rest content with successive mathematical subjects: first, completion of arithmetic; second, elementary algebra; third, intuitive geometry; fourth, simple trigonometry; fifth, demonstrative geometry. The latter plan follows an organization that is distinctly lacking in merit: the gradation is defective for much in algebra is distinctly simpler than certain topics in arithmetic. Similarly, introductory matters in geometry are less difficult than the later topics in elementary algebra. The usual college course in mathematics, consisting of advanced algebra, solid geometry, trigonometry, analytical geometry, and calculus, is now presented as an integrated course in mathematical analysis. The division of mathematics into its several branches has meaning for the adult mind familiar with the field of mathematics, but not for the novice who is often harassed by lack of understanding of fundamental concepts. The study labeled as a unified course in elementary mathematics is frequently little more than the old course with its isolated arithmetic, algebra, and geometry. In any one year the child studies in these ill-conceived and poorly integrated courses something in each of the three, but they remain three discrete sub-subjects. Although we have not achieved a generally accepted unified course in elementary mathematics, many of the unified courses seem to have proved themselves superior in the judgment of experienced teachers.

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[6] Experimental with reference to the learner, of course.

[7] "The Reorganization of Mathematics in Secondary Schools," *Report* of the National Committee on Mathematical Requirements (Mathematical Association of America, Hanover, N. H., 1923).

Reasons for including algebra in the elementary course. Many reasons have been offered to justify the inclusion of algebra in the elementary course of mathematics. Briefly, these may be summed up under five heads: (1) to aid pupils in analytical solutions by giving them mastery in the use of literal elements; (2) to aid them in interpreting industrial formulas; (3) to enrich the course in elementary-school mathematics; (4) to interest pupils in the work of the high school by giving them glimpses into secondary education; (5) to bridge the chasm between elementary-school and high-school curricula. To these Smith adds, (6) to train pupils in the logic of mathematics, and (7) to give pupils the ethical sense which mathematics can inculcate. The practice in European schools of giving more elementary mathematics that we do in the first eight or nine years has undoubtedly been a factor in broadening the scope of our work by the inclusion of elementary algebra.

Despite their merit these reasons are not sufficient to justify the teaching of algebra as a separate subject. Pupils should be given skill in using literal elements and in interpreting simple formulas, but this objective can be attained by teaching the simple equations of algebra and by making this an integral part of arithmetic. The course in arithmetic can be more properly enriched through varied applications to commercial and industrial life, rather than through condensing the course and adding a new branch of elementary mathematics. Whether or not teaching algebra in the elementary course will bridge "the dreaded chasm" and give a new interest in high-school work is extremely doubtful. The disciplinary and the ethical values of algebra can be derived from well taught arithmetic. Good elementary-school practice introduces the equation form very early. In simple number combinations the child becomes familiar with, $5+9=?$, $5+?=14$, $5\times9=?$, $5\times?=45$, $54\div6=?$, $54\div?=9$, etc. In learning to find areas he used $A=b\times a$, from which he is taught to deduce $b=\frac{A}{a}$ and $a=\frac{A}{b}$. In the following example the child is given A and a and must find b: "It was found that each child could be allowed only 12 square feet of ground for his individual garden. If the individual gardens must be 4 feet long, how wide are they to be made?" Substitute x for the interrogation point or for A or for b in the above illustrations and we have an algebraic equation. In finding interest the child learns $I=P\times r\times t$. When the principal, the rate, and the time are given, we need but change I to x, and, again, we have an equation in algebraic form. Uniform literal symbols and uniform solutions of simple equations having been taught, algebra will have made its most significant contribution to elementary arithmetic. In the chapter entitled "Problems" the details of the inclusion of algebra in arithmetic will be further considered and illustrated.

Geometry in the elementary-school course. Dissatisfaction with the methods of teaching mensuration in arithmetic led to the selection and simplification of a body of geometrical facts for the elementary and junior-high-school course. Children were too often taught to memorize definitions of geometric forms and formulas for obtaining areas and volumes through severely deductive lessons devoid of all concrete experience. The revolt against such unpedagogical practices led to a curtailment of this branch of arithmetic, to the gradual introduction of inductive lesson-forms, and to the correlation with manual training and drawing lessons. Thus was formulated that simplified geometry which is called *inventional* or *intuitive* or *experimental* geometry in which the laws of areas and volumes are derived by the pupils through useful manual activities.

It is very important not to teach all this geometric work in one grade, but to introduce it gradually and make it coextensive with the entire course in arithmetic. The great danger of assigning all this work to one of the upper grades is that it then becomes not a natural application of the fundamental processes of arithmetic, but a superadded, emasculated summary of the demonstrated geometry taught in the first or second year of the high school.

We shall present here in tabular form the character of the new course, but the actual content and illustrative lessons will be reserved for the chapter on denominate numbers (see pages 418 ff.).

Materials to Be Used: ruler; T-square; drawing board; triangles; compass; protractor; graph paper

Terms to Be Learned: distance; parallel; vertical; horizontal; oblique; perpendicular; degree; angle—right, obtuse and acute; square; rectangle; triangle; perimeter; base; altitude; area; volume or cubical contents; cube; parallelogram; circle; semi-circle; sphere; hemisphere; diameter; radius; circumference; pi (π); scale

Types of Applications: garden plots; cement walks; automobile runways; cementing garage and cellar floors; floor coverings; real-estate layouts and lot sizes; tiling; volume of rooms for estimating air capacities; water and gasoline tanks; making estimates of distances, areas, and volume; drawing to scale; reading and interpreting maps and scale drawings

We must stress again that these terms and processes are introduced incidentally when minds are intent on such manual activities as laying out, on the sand table, a neighborhood park, with appropriate drawings to scale, or making individual boxes of suitable size to hold drawing and writing media used by the pupils. It is not essential that the children be able to define the terms or give technically correct explanations of the various geometric forms. If learning occurs incidentally through actual manual participation, the concrete experiences will give adequate understanding of the basic concepts.

Industrial and commercial arithmetic. The elementary school must teach industrialized and commercialized arithmetic, not industrial and commercial arithmetic. Its function is to teach those facts of number relations and to develop those habits of manipulation that will insure intelligent reaction to any number situations. It is not designed to make a detailed study of the technical arithmetical practices peculiar to any industry. The specialized arithmetical needs of the electrician, of the builder, or of the import clerk must be met in the vocational training provided for these callings. Only arithmetic that relates itself to all the activities in the child's life, and that is common to all or most of the vocations in his environment is adequately industrialized and commercialized.

Problems for Study and Discussion

See end of Chapter III

CHAPTER III

THE ORGANIZATION OF THE COURSE IN ARITHMETIC

Our discussion has thus far concerned itself with the content of the course in arithmetic. We turn now to an equally important problem: how to develop and integrate the many topics of arithmetic. Theory and experience have led to the evolution of two very distinct plans of organization of subject-matter—the *ladder* and the *spiral*.

The ladder plan: exclusive attention to one topic for a considerable time. In the earliest courses of study each branch of arithmetic was like a ladder rung. Effort was focused on the mastery of one process in one grade or semester. The pupil felt secure on that rung before he mounted the next. In one grade the subject of notation and numeration was taken up in its entirety; in the next grade addition received exclusive attention. Similarly, subtraction, multiplication, division, fractions, denominate numbers, decimals, and percentage were taught successively, one by one, and completed, before the next subject or process was attempted. In the "addition grade" children added any quantities within reason. In the "fraction class" or classes, all manipulations with fractions were undertaken so that vigorous and persistent drill was assured.

The old schoolmasters justified this organization on many grounds. They argued that it gave opportunity for thorough grounding in essentials; that the organization provided for necessary drill; that concentration on a single line of thought was encouraged; that weaknesses in arithmetical knowledge of pupils could be traced forthwith to the proper grade, and thus responsibility could be fixed; that the child realized clearly that he was acquiring a new branch of the subject; that there

TABULAR SUMMARY OF PART OF A SPIRAL PLAN

	NOTATION AND NUMERATION	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION	TABLES OF WEIGHTS AND MEASURES	FRACTIONS	PROBLEMS	DECIMALS
1A	Counting objects by 1's through 25. Counting within same range without objects. Reading 1-place and 2-place numbers through 25.	Incidental, in the course of passing supplies, going on errands, etc.	Incidental, as in addition.			Concepts of more, less, higher, lower, longer, shorter, etc.			
1B	Meaning of number to be taught through 30. Counting objects 1's, 2's to 50. Counting without objects, 1's, 2's, 5's, 10's to 50. Reading 2-place numbers to 50.	Sign + and 17 combinations with 1, i.e. 2+1, 3+1, etc. and 1+2, 1+3, etc.	Sign − and the reverse of the additive combinations.			Measurement by boxful, basketful, handful, cupful, glassful, spoonful.		Such as arise naturally in activities of class and are within the number scope of the grade. All oral.	
2A	Counting with objects, 1's, 2's, 5's, 10's through 100. Reading 2-place numbers to 99. Reading 3-place numbers to 199. Writing 2-place numbers to 99.	*Oral* Sign = and 28 combinations, plus 2's and plus 3's. All combinations of 1A and 1B carried to 29, by decades. *Written* Double column, 2 and 3 addends, total not beyond 2 places. No zeros, no carrying. Single column, 3 and 4 addends.	*Oral* Sign = and the reverse of the additive combinations. Also minuend to be 29, answers to be 1, 2, 3, etc. *Written* Two-place numbers, no borrowing, no zeros.	Incidental as child learns number relations.	Incidental as situations arise naturally or as revealed in number relations.	Add concept of too large, too small, too high, etc. Cent, nickel, dime; pint, quart; taught through actual use.	1/2 of 1 and 2/2 = 1 developed objectively.	Of one operation and within the scope of the number work of the grade. All oral.	

		within the limits of the combinations of the grade.							
4A	Reading 5-place numbers through 99 999. Roman numbers through XII. Writing numbers through 99 999.	*Oral* Finding sum of 2-place numbers, no carrying, total not to exceed 100. *Written* Total limited to 5 places, number of addends not to exceed 5. Single column, number of addends not to exceed 9.	*Oral* Two-place numbers, no borrowing. *Written* Five-place numbers.	*Oral* Tables of 11's and reverses. Two-place numbers multiplied by 10. *Written* Multiplicand not to exceed 5 places, multipliers not more than 2 places. Product limited to 5 places. Dollars and cents.	*Oral* Reverse of multiplication. Three-place numbers divided by 10 with and without a remainder. *Written* Dividend not more than 4 places. Divisors, 11, 21, 31, 12, 22, 32, 42; no remainders, no zeros in dividend or in quotient.	Pound, half-pound, and quarter-pound.	*Oral* Halves, quarters, eighths; thirds, sixths; fifths, tenths. 2/2 = 3/3 = 4/4 = 10/10 = 1. 1/2, 1/4, 1/8, 1/3, 1/6, 1/5, 1/10 of multiples of 2, 4, 8, 3, 6, 5, 10 respectively.	One operation. Limited to scope of numbers taught in the grade.	Dollars and cents.
6A	Reading and writing numbers of 4-place decimals.	Systematic practice to promote accuracy and speed.	Systematic practice to promote accuracy and speed.	Systematic practice to promote accuracy and speed and familiarity with all types. Remedial drills based on results of diagnostic testing.	Same as for multiplication. Remedial drills based on results of diagnostic testing.	*Oral* Table of U.S. Money. Use of 1/2, 1/4 inch in measuring lines and small common objects. *Written* Interpreting simple scale drawings; application of scale to maps in geography. Line graphs of pupils' progress.	*Oral* Mixed number to improper fraction; integer not to exceed 9, denominator not to exceed 10. Reverse, change improper fraction to mixed number. Multiplication of fractions by fractions and finding fractional part of a fraction. Finding what fractional part one number is of another. *Written* Addition and subtraction. Number of addends not to exceed 6. Same as above (oral work).	Within limits of the formal work of the grade. One and two operations. Include problems without numbers.	Addition and subtraction, 4 decimal places, number of addends limited to 6. Multiplication of decimal by decimal, the number of decimal places in answer limited to 4. Effect on decimal of multiplying by 10, 100, 1,000.

was a definiteness of assignment too frequently absent in present-day courses.

That there are advantages in the course organized on the ladder plan must be admitted by its opponents. But its inherent weaknesses are very serious. Exclusive attention to each of the separate and succeeding branches of arithmetic tends to mechanize the subject. The richness derived from a course which contains variety of application is lost. The various branches of arithmetic are mutually supplementary, and, when properly interrelated, contribute towards clear comprehension. Under the ladder plan of artificially partitioning the subject, the child did not realize that 5/6 and .83⅓ per cent have the same value; that decimals are fractions and that percentage and decimal forms are similar. It is obvious that the subject as a whole must be taught as interrelated functions, and that its component parts must not be isolated.

This system of exclusive attention to each topic in arithmetic is responsible for lack of intelligent gradation. In a general way, it is true that subtraction is more difficult for the child than addition and that multiplication ranks next in the scale of difficulty. But in the development of these processes, this succession of addition, subtraction, multiplication, division, represents no gradation. It is much simpler for the child to learn to subtract 32 from 78 than it is to add ten numbers of five orders each; it is simpler to multiply 23 by 3 than to subtract 243 from 702. Similarly, it is more intelligent gradation to teach the child to divide 936 by 3 than to postpone this type of example until he can multiply 7394 by 608. There is no reason to justify the teaching of all the cases and the forms of one process before taking up the next. A vivid realization of these limitations in the method of exclusive attention to each branch of arithmetic led to the modern plan of the course of study.

The spiral plan. Under the more modern plan of organization the various branches of arithmetic are taught simultaneously and, spiral fashion, are repeated in successive grades, but in ever-widening circles. A helpful illustration of this type of spiral organization is found in an analysis of the revised course of study for the schools of the City of New York. Let the reader select any one topic in arithmetic—say addition or common fractions or decimals—and trace it through the grades, vertically. He will then see that it is not taught completely in any one grade, but is repeated through several grades and always in progressively widening circles as the course grows. By tracing the total grade assignment horizontally in the accompanying tabular summary of a spiral course, he will perceive at once the variety of work taught in any one semester and the limited number of skills which are developed in any one branch.

Figures 1, 2, and 3 show how one topic in arithmetic—common fractions, decimals, or percentage—may be distributed over a number of grades. The student of educational practices who is concerned with the justification for any particular type of assignment is doomed to disappointment because he will find not only disagreement among makers of courses of study, but the unchallenged rule of personal or

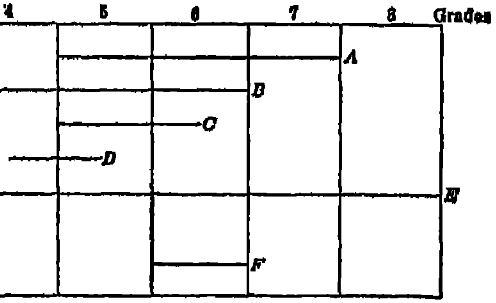

FIG. 1.—GRAPHIC REPRESENTATION OF GRADES IN WHICH COMMON FRACTIONS ARE TAUGHT.

Line *A* represents a school system which teaches fractions in three years, line *B*, in three years, but not the same years; line *C*, 1½ years; line *D*, in 1 year (from middle of the 4th year to middle of 5th); line *E*, in 5 years; line *F*, in 1 year.

Courses represented by *A*, *B*, *E* are very spiral in development; courses represented by *D* and *F*, are practically opposite to spiral forms.

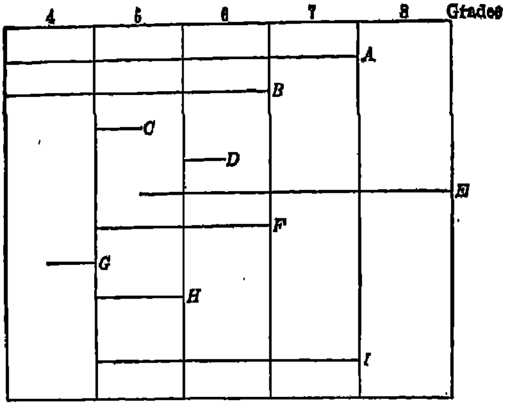

FIG. 2.—GRAPHIC REPRESENTATION OF GRADES IN WHICH DECIMALS ARE TAUGHT.

Lines *A*—*I* represent different school systems. Courses represented by *A*, *E*, *I* are evidently spiral; by *B* and *F*, moderately spiral; by *C*, *D*, *H*, opposed to spiral plan.

Note differences: Course *G* teaches decimals in second half of the 4th year; *C*, in first half of 5th year; *D*, in first half of the 6th year. Course *A* distributes the work over 4 years; courses *C*, *D* and *G*, limit the work to only one-half year.

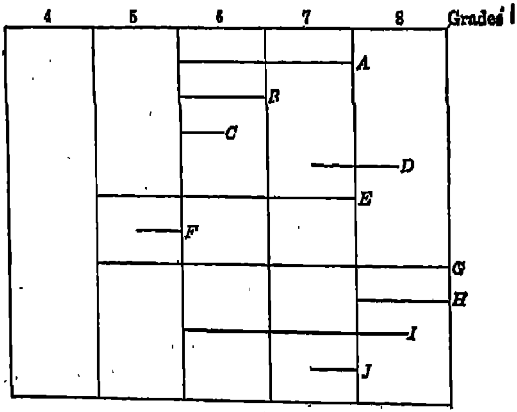

FIG. 3.—GRAPHIC REPRESENTATION OF GRADES IN WHICH PERCENTAGE IS TAUGHT.

Lines *A*—*J* represent school systems. Point out differences in time required by different systems. Which are extremely spiral? Moderately spiral? Opposed to spiral plan? Compare courses *F* and *J*, courses *F* and *G*.

local opinion in allocating the various branches of arithmetic to the several grades.

Reputed advantages of the spiral plan. The advocates of spiral courses in arithmetic set forth many advantages for them. They argue: *a.* Pupils who leave school before the completion of the eighth year are introduced to a richer and a wider range of topics in arithmetic. The arithmetic of practical life presents a marked diversity of topics and processes.

b. Greater correlation of the various branches of arithmetic is made possible, for in one grade the child learns that common fractions, decimals, and percentage are mathematically the same despite their difference in form. Children learning to add and to multiply fractional quantities apply these processes at once to business forms, mensuration, and denominate numbers. Such correlation affords natural and varied applications.

c. This correlation leads to more rational associations, and therefore to better comprehension and more lasting impressions.

d. All spiral courses are replete with opportunities for reviews of the work of preceding grades. The child who could not master the initial concepts of decimals in Grade 5B has not permanently lost this branch of arithmetic, for decimals will be taught again in 6A, and the 5B work will be reviewed as an apperceptive basis for the lessons in the higher grades. In the final analysis, the spiral plan makes for distributed learning.

e. Another important advantage in a spiral system is found in the fact that the subject is better graded, is more psychological than logical, and is therefore better designed to insure full mastery. As the child grows older and gains in mental power each of the various branches of arithmetic gradually increases in content and in difficulty. In the older system, in which fractions were completed in one grade or in one year, the young child was naturally required to learn the most difficult processes in fractional computations. But many children were not mentally matured for such thoroughness. In the spiral course fractions may be taught in three or four grades; only the simplest concepts of fractions are developed early in the school course, and as the semesters progress, the necessary *facts and generalizations* are *introduced in a* graded development that parallels the child's increasing maturity. To attain these reputed advantages many curricula in arithmetic are organized on severely spiral lines.

Limitations of spiral courses. An extremely spiral course often suffers from the defects of its own virtues.

a. Incidental reviews and drills in correlated subject-matter are always helpful, but when these teaching practices are employed beyond all reasonable needs they become drags on teaching. Superior children, we are told, lose interest in the subject because they do not need what, to them, seems a riot of repetitions. They see too little new subject-matter in successive grades.

b. Lack of unification of subject-matter is often another danger of the spiral unfoldment. Many teachers are not familiar with the entire course of study, and fail to see the assignment of their respective grades in relation to the whole subject. They unwittingly set the whole spiral system aside. Thus, teachers of fifth or sixth-year classes often teach numeration and notation in the early part of the year, drill in fundamentals in the second part, fractions in the third part, etc., and thus revert to the system of exclusive attention to one topic, not for a whole grade, but for part of it. Other teachers try to cover the entire assortment of topics each week. On Monday notation and numeration and fundamental processes of whole numbers are taught; on Tuesday fractions take the whole lesson; on Wednesday, decimals; on Thursday, problem work; Friday is devoted to a grand review of the week's experiences. Obviously, neither of these two plans possesses merit. Inept and inexperienced teachers find it exceedingly difficult to plan their grade work when assignments are so varied.

c. Indiscriminate correlations create at least as many difficulties as they solve. Interrelations do not necessarily clarify a situation. To a student who has a bird's-eye view of a subject and an elementary knowledge of its basic concepts, interrelations give new viewpoints and deeper grasp. But to his classmate who is merely threading his way through a new experience, these very interrelations may become sources of confusion. The child who has an adequate foundation of the principles of common fractions, decimals, and percentage, and has had varied practice in their application, usually experiences no difficulty in thinking of 83⅓% as "⅚" or of 200% as "2 times." These are different aspects of the same relationship. The pupil whose contact with common fractions is limited to a single semester cannot follow the explanation of a fraction as ratio; he does not see that ⅔ refers to "2 out of 3." The inexperienced are not ready to perceive new relationships and to formulate new generalizations.

d. Severely spiral courses of study make difficult the fixing of responsibility in a large school. When any one teacher presents six or eight branches of arithmetic, introducing some for the first time, and carrying others already met in an early grade to higher technical levels, the supervisor may not be able to trace weaknesses to their sources.

The modified spiral development. Theoretical analyses and practical experience point to the need of evolving a modified spiral course of study in which the spiral characteristic is basic throughout the course, but in which each grade stresses certain branches of arithmetic so as to mark them as the distinctive contribution of that semester. There is every reason for retaining the spiral system in the earlier grades. But beginning with the intermediate grades (fourth year), opportunity must be given to focalize attention on fewer topics and processes so that drills may be more intensive. The severely spiral course presents a merry-go-round of topics in each grade, and invites a feather-duster treatment of each. In the modified spiral system it is possible to subject each topic to careful study so that thoroughness of comprehension and permanence of knowledge may result.

Conclusions. We have no body of objective data, scientifically derived, to settle the issues here outlined. We do not really know the comparative difficulty of the various arithmetical operations, and are therefore uncertain about their assignment to the various grades. As a rule, the introduction of an arithmetical topic in a grade is determined by the demands of other subjects of the curriculum, or by the textbook that the children use, or by the book that the teacher follows, or, too often, by mere tradition—by almost any factors except those that should prevail in a scientifically devised course of study.

The Time to be Devoted to Arithmetic

Obvious questions of considerable importance concerning the time to be spent on arithmetic present themselves to those making a course of study. How much of the total school course should go to arithmetic? What part of each day or each grade ought the subject receive? How long shall the lessons be? Is the afternoon session less productive of results in arithmetic than the morning session? When shall formal instruction begin? Naturally we seek answers in terms of objective data.

Relation of time to proficiency. A study of the time allotted to arithmetic in typical school systems reveals one interesting fact—the widest range of variations. Obviously there are no scientific data to guide us in determining approximately what part of the school time to devote to arithmetic. *A priori*, it would seem that the more time given to the teaching of arithmetic, the greater must be the proficiency developed by the pupils. But investigations made under experimental conditions do not warrant such an inference. They seem rather to indicate no relationship between time expenditure and the results of instruction. If we were to select at random a school from a list of schools that give more than average time to arithmetic, the chances are about equal that we would have chosen a school where the work is below standard, and, conversely, if we were to select a school from among those that devote less than average time to the subject, we should probably find the children doing superior work in it. It was found that schools attaining a rating of 80 per cent in arithmetic and those scoring only 25 per cent in the same tests gave the same time to the teaching of this subject.[1]

The general drift of these findings is somewhat disturbed by the conclusions of Rice.[2] He found that there was positive correlation between efficiency in arithmetic and time devoted to the subject, but that the correlation was not large, only .365; that within any one school system there seemed to be no relation between the standing of a school in arithmetic and the time devoted to its study.

The following tabulation gives a general idea of the practice in American schools with reference to time allotment to arithmetic. It seems reasonable to conclude that about 10 per cent of the available time is taken by arithmetic.

Time Allotted to Arithmetic

Average Number of Minutes per Week Devoted to the Teaching of Number Work *

Reported by	Grades					
	I	II	III	IV	V	VI
50 cities in 1914	71	149	203	231	223	226
42 cities studied by F. C. Ayer about 1924	96	143	193	206	211	211
52 cities in N. J., 1925	97	165	216	231	228	220
444 cities in 1926	80	146	106	211	215	215

* For bibliographic data and for figures from which this tabulation is made, see pp. 75–76 of the *Twenty-Ninth Yearbook* of the National Society for the Study of Education.

More recent studies bear out the results of these older ones. The time devoted to arithmetic is not among the most important factors determining efficiency in this subject. The constructive character of the supervision, the wise use of diagnostic tests and prompt application of remedial and differentiated drills, the rational motivation of drills, the stress on real problems—these hold greater promise of growth in arithmetic than the mere matter of gross time allowed to this subject.

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[1] C. W. Stone, *Arithmetical Abilities and Some Factors Determining Them,* Contributions to Education, No. 19 (Teachers College, Columbia University, 1908).

[2] J. M. Rice, "Tests in Arithmetic," *Forum,* Vol. 34, November and December, 1902, pp. 281–297.

Length of teaching period in arithmetic. Questions concerning length of teaching periods invariably revert to discussion of fatigue, a condition as little understood in education as shock is in medicine. Experience and the composite judgment of teachers seem to point to the following limits of teaching periods in such subjects as arithmetic: in the first two years, ten to fifteen minutes; in the third and fourth years, fifteen to twenty minutes; in the fifth and sixth years, twenty-five to thirty minutes; in the early junior-high-school grades, thirty to forty minutes; in the later junior-high-school grades, forty to forty-five minutes. While it is wise not to prolong an arithmetic lesson beyond the time limits we have set, modifying conditions arise which may insure good work in protracted periods. Genuinely motivated work will enable the child to apply himself to a task for periods twice as long as we have indicated. Supervised learning of arithmetic by a laboratory procedure and the introduction of manual activities into arithmetic will achieve the same ends. The writer came upon a fifth-grade pupil in a Dalton School who had been at fractions for about two hours and showed no desire to change to another subject. When asked the reason for this attachment to arithmetic, she explained, "This is the first time I really understand what these denominators mean; now I can change to mixed numbers, and add and subtract and do what I like with fractions." According to the traditional laws of fatigue, two hours of arithmetic should have produced complete exhaustion. Under certain adverse circumstances the development of fatigue is accelerated so that children are brought to a state of exhaustion in one hour; under more favorable circumstances, there is a decided retardation in the rate of development of fatigue.

Relative effectiveness of morning and afternoon sessions. With the development of the platoon plan of school organization and the departmentalization of instruction, the question of the relative effectiveness of morning and afternoon sessions becomes more persistent. Tradition has decreed that arithmetic be taught early in the morning session. Is it not the most difficult subject in the curriculum? Should it not receive the most favorable period of the school day when minds are most rested and ready, therefore, for the most exacting tasks? Unfortunately, the answers are not as simple as these questions imply. Arithmetic is undoubtedly the most difficult subject to some pupils, but, surely, not to all. Some parts of the subject fail to interest even those who are proficient in it. Genuinely motivated work is often done very well late in the morning session and in the afternoon session. The time of the day is only one factor and, generally speaking, not the most important factor determining the quality of one's performance.

Thus far experimental studies point to the conclusion that for arithmetic the afternoon session is only slightly less desirable than the morning session; that this preference does not hold for all children; that for some children the afternoon session is decidedly less favorable. These conclusions are far from final because of the number of conditions that were not factored out in the experimental studies. The selection of socially vital experiences and the introduction of effective teaching practices will usually counteract the diminished adaptiveness of the mind in the late periods of the school day.[3]

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[3] For more detailed accounts of these studies, the reader is referred to the following:

J. M. Glass, "Curriculum Practices in the Junior High School and in Grades 5 and 6," *Supplementary Educational Monographs*, No. 25 (University of Chicago, 1924).

W. H. Winch, "Mental Adaptations during the School Day as Measured by Arithmetical Reasoning," *Journal of Educational Psychology*, Vol. 4, January and February, 1913, pp. 17-28, 71-84.

Studies on fatigue frequently discuss the question but they do little more than skirt around it.

When shall systematic teaching of arithmetic begin? *A plea for later introduction.* There is a growing sentiment among intelligent laymen, as well as in progressive and child-centered schools, for the postponement of all formal teaching of reading, arithmetic, spelling, and penmanship. The reasons are not especially new nor are they based on anything more accurate than personal opinion. Objective tests given by one investigator to children in the upper grades seem to indicate that systematic number lessons may well be "deferred until the end of the third year of the school without placing the student at a disadvantage in relation to the acquisition of skill in fundamentals.[4] We cannot accept this conclusion because another group of studies leads to a contrary decision. "The evidence is now sufficient, it is believed, to justify a decision to teach regular—perhaps formal is not the word—number work in the first grade."[5] In the light of these conflicting findings we have no adequate data for eliminating regular instruction in arithmetic from the first three years.

Before the discussion proceeds we must define our concept. By formal or systematic teaching of arithmetic we mean the direct attempt to make automatic the handling of certain number combinations and the recognition of number relations. Surely we do not mean keeping number experiences out of the child's life, for that cannot be done.

Those who argue for delay in formal teaching of arithmetic urge the introduction of a rich experiential background for later number work. Children who handle things, who plant, who feed the classroom menagerie, who cut and fit materials, who play in groups, and who are responsible for the distribution and collection of classroom equipment, are constantly coming into varied relations with quantity. An intelligent, illiterate person exhibits considerable understanding of number.

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[4] Garrison and Garrison, *The Psychology of Elementary-School Subjects* (Johnson Publishing Co., 1929), p. 433.

[5] B. R. Buckingham, "The Social Value of Arithmetic," *Twenty-Ninth Yearbook* of the National Society for the Study of Education (Public School Publishing Co., 1930), p. 51.

Life, not formal schooling, taught him his arithmetic knowledge. Hence a rich background of well directed experiences in the introductory grades will do much to quicken formal learning of number in later grades.

The plea for delay continues: Children will probably learn the fundamentals of arithmetic better when they are two or three years older. With increasing maturity, the pupil perceives the need of arithmetic skill, and learning is thus aided by real motivation. To postpone systematic learning and supplant it by informal learning promises hygienic gains for most children. Here we have psychological, pedagogical, and social arguments, though no proof, for the exclusion of formal arithmetic from the first two years of the school course.

The case for early systematic teaching of arithmetic. In the absence of conclusive proof a tenable case can be made out for the opponents of delayed introduction of arithmetic. They express the fear that informal teaching will resolve itself into no teaching, and that the child will abstract from his concrete experiences number ideas that are vague and perhaps inaccurate. Since the introductory phases of arithmetic are concerned with acquiring formal skills, not with comprehension, it seems wise to let the child form his basic habits in addition, subtraction, multiplication, and division as early as he will accept them. From some children we cannot keep number experience. But when these children work without supervision they may acquire a set of undesirable number habits that involve undue dependence on crutches of their own invention. Unfortunately, the period of school education is deplorably short for many youngsters. For them, number facts and number skills must, of necessity, be acquired very early. And, finally, we are reminded that normal children of seven show conclusively that they can master the basic number concepts usually taught in the first two grades.

Early instruction in arithmetic need not be formalized. But it must not be assumed that because arithmetic is made an integral part of the work of the first two years that it is advisable to return to the deadening grind of the mechanical drills in additive combinations or in multiplication tables. Modern psychology forbids such procedure; modern methodology condemns such practice. Although the work of the earliest grades is definitely and formally prescribed, it may be enlivened in many ways. Every number fact or process must be motivated by showing its social worth or by relating it to the instinctive life of the child; throughout the development of the early lessons the inductive method must prevail so that the child is the active discoverer of knowledge; every drill must be presented through some form of physical activity so that it appeals to the senses of the child. Arithmetic so taught need not be devoid of real interest.[6]

Arithmetic in the kindergarten. It is commonly believed that children of four or five are usually too young to grasp number facts. Although no attempt should be made to give formal instruction to these children, certain mathematical knowledge, as we saw, cannot be kept from them. Their work and their play call for considerable arithmetical activity, for example: counting to twenty and even to thirty, recognizing certain geometric forms and small groups of objects, adding and subtracting small quantities, and using halves and even quarters. But all these arithmetical facts should be taught, not by consciously setting out to add them to the children's stock of knowledge, but incidentally, in the round of pleasurable activities that should make up their lives.

PROBLEMS FOR STUDY AND DISCUSSION

1. The quality of instruction in any subject cannot rise above the character of the material prescribed in its course of study. Discuss with reference to the teaching of arithmetic.

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[6] This question is discussed helpfully in the following references: W. A. Jessup and L. D. Coffman, *The Supervision of Arithmetic* (Macmillan Co., 1916).

J. S. Taylor, "Omitting Arithmetic in the First Year," *Journal of Educational Administration and Supervision*, February, 1916, pp. 87–93.

2. Criticize the following as grade assignments in courses of study:

   (*a*) Develop habits of accuracy in this grade.
   (*b*) Teach the use of short cuts.
   (*c*) Develop speed consistent with accuracy.
   (*d*) Teach two and three step problems.

3. If Rousseau had been asked what arithmetic to teach children of school age, he would have answered, in all probability:

   (*a*) What the child will need most when he grows up.
   (*b*) What he thinks he needs now.
   (*c*) Teach him no arithmetic; leave the task to life.

Which of the three statements would you ascribe to Rousseau? Why? Substitute for Rousseau, other educational leaders like Pestalozzi, John Locke, Horace Mann, or John Dewey. Which statement expresses the position of each of these? On what facts do you base your answer?

4. Formulate course-of-study assignments for each of three levels of pupil ability in the following topics: (*a*) division by a common fraction; (*b*) reduction of common fractions to percent equivalents; (*c*) finding the cost of a quantity (expressed by a mixed number, 3½) after the cost of a fractional part is known (what will 3½ cost if ⅜ cost 9 cents). What part of Chapter II will help you in this task?

5. Plan a research project designed to reveal what is most useful for children to learn in arithmetic. Phrase your problem carefully, indicate the plan of the inquiry, the method of obtaining data, the cautions to be observed, the method of checking the reliability of your procedures and of the result, and some of the difficulties you would meet.

6. List the arithmetical activities that you must perform in the course of a typical day or a week. Compare your apparent need in arithmetic with the content of a course of study in arithmetic. What difference do you note? Make two reasonable inferences from the comparison.

7. The text gives six ways of making provision for individual differences of pupils. Supply for each of these methods a very definite illustration. State age and grade of the pupils you have in mind and the specific objective of each lesson.

8. Analyze a textbook in elementary mathematics designed for the junior high school. Are the branches of mathematics genuinely integrated or do they follow in succession? Support your answer by specific references to the book or books.

9. Secure a course of study in arithmetic with which you are not

familiar. Analyze it to determine whether it is organized on the *ladder* plan, the *spiral* plan, or *modified spiral* plan. Justify your findings. What changes in assignment of topics to grades are you tempted to make? Why? What kind of justification have you for the proposed change, scientific or personal opinion? Make the same kind of analysis of an arithmetic textbook.

10. A debate is an attempt to triumph, not to establish a truth. Refer to three important problems of curriculum organization that illustrate the truth of this statement.

PART III

PSYCHOLOGICAL CONSIDERATIONS INFLUENCING THE TEACHING OF ARITHMETIC

CHAPTER IV

THE DIFFICULTIES THAT BESET THE LEARNER

Dissatisfaction with the results of instruction in arithmetic. Few complaints against current teaching come with greater frequency than those directed against the teaching of arithmetic. It is charged that the graduate is inaccurate, unduly slow in his calculations, and unable to meet a new arithmetical situation, even if only slightly varied from the type. To answer the indictment, school superintendents have analyzed the charges and have ascertained the amount of arithmetical knowledge and skill which representative business men require of our elementary-school graduates. Invariably the total requirements are few and simple when we consider the extensive course in arithmetic and the high standards of proficiency prescribed in it.

Two firms that are among the largest employers of school graduates in New York City presented the following as specimen tasks which they expect young people to perform accurately:

1. Addition of sums like,

$$\begin{array}{r} \$1.58 \\ .65 \\ .49 \\ 2.98 \\ 10.77 \\ 1.38 \\ 3.10 \end{array}$$

2. Multiplication, fractions, addition, filling out a sales slip, computing the values, and adding the amounts as shown below:

Bought by Henry Wise, 649 Classon Ave., Bklyn.

Sales No.	Carrier	Charge	Date
168	15	Ch	5/16/33

Quantity	Items	Amount
2¾	White lace @ 57c.	
3	Spools white silk @ 10c.	
1¾	White silk @ 93c.	
		Amount

These two specimen tasks were given to members of Grade 8B in one of our largest city school systems. Of the 1023 who tried the addition example, only 76 per cent were correct; of the 962 who did the second, only 66 per cent were correct. The lowest class record in the first was 68 per cent; in the second, 42 per cent; the highest class record in the first was 91 per cent; in the second, 92 per cent. There is, then, a startling difference between the standards which teachers set for their pupils and the actual achievements; between the modest demands of business and the demands of the school itself. How can we account for the gap between endeavor and achievement?

Causes of failure in arithmetic. Every circumstance that has ever accounted for any failure is presented as the sole cause of failure in arithmetic. That there is no one cause in so complex a situation is obvious. The causes are numerous and may be grouped into three classes: (*1*) inadequacies of the pupil; (*2*) inadequacies of teaching; and (*3*) inadequacies of supervision.

1. *The pupil is not always equipped to master the complex techniques of arithmetic.* His native intelligence may be limited, his vocabulary may be woefully meagre, or his knowl-

edge of common business practices may not extend beyond simple store purchases. But none of these may be the causative factors in a given case, for we find that some children who fail in arithmetic have high intelligence quotients, or give evidence of genuine ability in reading or in the social studies.[1] And yet, arithmetic is responsible for a greater number of failures than any other school subject.[2]

The explanation often takes a different form. Arithmetic, we are told, is a subject of cumulative and progressive skills. The loss of one branch of the subject makes impossible success in those branches that follow. Which of us in teaching percentage does not lay the failure of some of our pupils to their ignorance of decimal fractions? Still others argue that the difficulties experienced in language, in reading, in social science or in the manual arts are largely objective, and are revealed at once in the very responses made by the children. But how different in arithmetic! Here the child struggles with his number difficulties in silence; the struggle is subjective and internal. When the answer is correct, we assume that the challenge of the example has been met through planned and intelligent execution; when, on the contrary, the answer is incorrect, we make the contrary assumption. But the child's explanation of a solution may deceive us completely by its correct repetition of set phrases, well memorized, but little understood. A teacher was completely baffled by one of his pupils who invariably got the wrong answer to type III in percentage despite his correct statement: "If the milkman who delivered 560 bottles of milk, served 7/8 of his customers, then 1/8 of his customers is 1/7 of 560, which is 70." The child's vocal apparatus learned to say what is considered correct; but his limited understanding of the situation told him "1/8 is not 1/7" and he invariably computed 1/8 of 560 instead of 1/7 of 560.

The nature and the variety of the relationships among quantities set up many interferences in learning. We have become so accustomed to these situations that we take them for granted. But to the immature pupil who is not gifted with superior intelligence arithmetic seems rich in apparent inconsistencies. To illustrate: Multiplication connotes to these children, "Make larger." Is not multiplication only a short and convenient form of addition? But 8×½ gives 4, an answer that is decidedly smaller than the original quantity. One half times 8 means, of course, not all of the 8, but only half of it. But "½×" produces no more than "½ of." The children may find this puzzling. 54×1 produces 54, a product even less than the sum of 54 and 1. Similarly, they assume that the process of division always yields smaller quantities. 8÷2 gives 4—a result that seems reasonable. But 8÷½ produces an unexpected increase to 16. In ½+¼, we obtain not 2/6 but ¾. In ½×¼ the result is ⅛. To many pupils the puzzling questions, therefore, are when should numerators be added, when should denominators be multiplied, and why should multiplying yield a smaller result than adding?

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[1] C. H. Judd, *Measuring the Work of the Public Schools* (Russell Sage Foundation, 1916), pp. 26–29.

[2] *Ibid.*

Subtraction does not seem a consistent process to many children. Let us see what they find baffling in two different subtraction examples.

From 4 bu., 1 pk., 2 qt.	From 534
Take 2 " , 2 " , 1 "	Take 147

In both borrowing is involved. A pupil changes 1 bushel into 4 pecks and now has 5 pecks from which he takes 2 and has 3 pecks left. In the second example 7 units cannot be taken from 4 units. A ten is converted into units and *10 units* are obtained. These are added to 4 units, making 14 units from which 7 can now be taken. To us, the two forms of reducing a higher denominator to a lower (one bushel to pecks and one ten to units), must, of necessity, give different numbers of the smaller denominations. To the learner this may be inconsistency itself, and, therefore, sets up learning conflicts.

The lesson taught by these illustrations is obvious. Progress must be slow and children's difficulties must be met understandingly as well as patiently.

2. *The cause of failure may be found in the technique of unskillful and inexperienced teaching.* Ungraded subject-matter presents too many difficulties at one time. Unmotivated teaching may fail to arouse attention. An indefinite lesson aim may fail to focus pupils' minds on the specific objective of the lesson. Absence of necessary objective aids renders simple statements empty generalizations. Failure to provide for varying capacities of children leaves most pupils uninfluenced by the teacher's efforts. Taught as a succession of procedures without reference to any of the activities in the child's life, arithmetic becomes an unrelated complex of skills. Practice without focused attention on the specific weakness of individual pupils does little to establish habits in arithmetic. The list of teaching errors may be increased to a startling number.

3. *To inexpert supervision* we may ascribe many of the difficulties which beset children in learning arithmetic. Among the improper conditions which we charge to inadequate supervision we may list the following: courses of study cluttered with outdated arithmetical practices and problems; the use of textbooks poorly conceived and poorly written; failure to provide the best printed drill material; infrequent visits to classrooms and less frequent conferences on methodology with teachers; inadequate equipment of illustrative materials and visual aids to instruction; lack of use of standardized tests to discover the true measure of children's attainment by comparison with norms of children of like mental age and grade; neglect of diagnostic testing to reveal the specific weaknesses of individual children. Supervision that is sympathetic and constructive helps to make the arithmetic lesson a stimulating succession of challenges to teachers as well as to children.

Our introductory discussion indicates that this section will study in detail a series of teaching principles distinctly psychological in their implications, which aid in devising measures for the removal of many causes of failure in arithmetic.

PROBLEMS FOR STUDY AND DISCUSSION

1. Secure the privilege of observing three or four people who are deficient in arithmetic. Which of the causes listed under "Deficiencies in the pupils" seem to account for the trouble that these children are experiencing? How would you plan to help the children overcome the difficulties you observed? What learning conflicts were confusing these children?

2. List the ineffective teaching practices that may account for pupil failures in arithmetic. Illustrate each of them, and indicate for each a more desirable procedure. Wherever possible, state what psychological principle was violated in the ineffective procedure.

3. A pupil who is deficient in arithmetic has been referred to you for help. How would you plan to discover (*a*) the specific weaknesses of this pupil, and (*b*) the probable causes of the deficiencies? Run through the items of the index for aids in formulating your procedure.

4. Prepare a bibliography of six items on causes of pupil deficiency in arithmetic. Consult the bibliography at the end of the book.

CHAPTER V

THE LANGUAGE AND THE ALGORISMS OF ARITHMETIC

The language of arithmetic. Every body of knowledge specialized enough to be designated as a separate subject has a vocabulary peculiarly its own. It is assumed that the child will learn the meaning of these technical terms incidentally through their repeated appearances in successive situations. In a measure this assumption is justified. But, like all indirect and incidental learning, much needless difficulty is encountered. It is the part of wisdom to anticipate these difficulties of language by explaining each new term as it arises in its natural setting.

Children in Grades 5 through 8 were asked to give the meaning of *area* as used in the examples they were solving.[1] Among the many which Stevenson found are the following:

> The sum you get when you add different lands, 1 acre + 2 acres = 3 acres; length and width added together, 2 ft. long and 3 ft. wide = 3 ft. = area; number of miles; width of a field; number of feet in a mile; number of people that live in the United States; the size of anything, 24 ft. + 28 ft. = 52 ft.; the area is 55 feet below the sea level; it is the boundary of a country.

Each teacher took for granted that the word *area* carried the correct meaning for each child, and no teacher ever stopped long enough to give the word its true connotation.

Confusion often arises because teachers fail to realize that a familiar word may be used in a peculiar sense in arithmetic. The expression, *How many*, usually carries the idea of much, of quantity, and seems to invite increase or addition. Nevertheless many examples ask, "How many marbles did John have left if he began his game with 24 and lost 9?" Clearly, the subtractive idea of *how many* is new.

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[1] P. R. Stevenson, "Difficulties in Problem-Solving," *Journal of Educational Research*, Vol. 11, February, 1925, pp. 95–103.

Most new books use such terms as *quotient, difference, product,* and *result.* In actual practice we say, "Find what was left," not, "Find the difference"; "Find how many books I can buy," not, "Find the quotient." Economy of time and other considerations force these terms upon us. But it is important that we recognize that these words are technical and as unfamiliar to children as the terms *delta, tributary,* and *plateau* met in their geography class. We are not for the elimination of useful technical terminology, but for its gradual introduction with due attention focused on meaning. Almost every new procedure in arithmetic requires an introductory language lesson.

Thorndike reports[2] that almost all of the following words and expressions are found in the first fifty pages of arithmetic textbooks commonly used for beginners:

> Area, allowance, average, balance, bin, contained in, difference, duty, dealer, debt, deposited, due, equal, earning, gain, gross, have left, invoice, income, installment, less, loss, lose, measure, margin, net, per, produce (what number multiplied by 3⅓ will produce 13), profit, purchase, retain, receipt, taxes, total, value, and whole.

Small wonder that some children believe that an average is "What a chicken lays its eggs on." How frequently do their textbooks state: "A brood of chickens laid 98 eggs in a week. How many eggs did they lay on an average each day?" What is more natural for the children than to assume that an average must be a convenient shelf on which these obliging hens deposit their gifts to mankind?

We have evolved a variety of ways of calling for addition and subtraction. Judd[3] has conveniently listed these for us:

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[2] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), pp. 90–91.

[3] C. H. Judd, "Psychological Analysis of Fundamentals of Arithmetic," *Supplementary Educational Monographs,* No. 32 (University of Chicago Press, February, 1927), pp. 81–83.

Add: Find the sum. Find the total. Find the amount. Find the value. Find the cost. Find the distance. Find the whole. Find the length. What is the sum (total, amount, value, cost, distance, whole, length)? How many? How many in both? How many together? How many in all? How far? How long? How old? How much are they? What is the combined cost (distance, length)? What is the price?

Subtract: Take away. Find the remainder. Find the balance. Find the difference. Find what is left. How many more? How much longer? (taller, heavier, farther)? How many are needed? How many are lacking? How many others are there? Not counting ———, how many more are required?

At times the difficulties of language are intensified by the peculiarities of the situation demanding arithmetical solution. The teacher of arithmetic will be repaid for the expenditure of time required to analyze each of the following examples selected from class assignments and textbooks:

1. How much must I buy? I have 17 marbles and need 19. Many children could not understand why a boy should buy 2 marbles when marbles are won, not bought, nor why anyone should need just 19 marbles.

2. At what time will Bob reach home? He started at one o'clock for the lake, spent two hours fishing, and one hour each way in travel.

The children had no difficulty in adding 1, 2, and 1. Some said, "four" and gave the answer, "Four o'clock." Others were baffled by the expression, "spent two hours fishing" because they invariably associate *spent* with *money*.

3. Four children are making furniture for their play-house. They decide to make 7 chairs. After the fifth chair is completed, they must make how many more?

Note the difficulties:

a. Four children. Why four? Why confuse the situation by telling how many children?

b. Visualizing the situation, for example, what are the children trying to accomplish? How far have they carried the plan? What remains to be done?

c. The use of comparative subtraction: 7 is how much more than 5? or 5 is how much less than 7?

d. The introduction of the ordinal, *fifth*, for the quantity, *five*.

4. John is 12 years old; Mary is 9. In how many years will Mary be as old as John now is?

To the child, subtraction means, make less or take away. Here a number of years must be added to Mary's age to make 12. Although the arithmetical process is simple enough, the total situation is more difficult than we suspect.

Interesting studies[4] have been made of language difficulties experienced by children in arithmetic. These yield additional data to support our insistence on attention to the language of arithmetic. Brooks[5] analyzed five books designed for the third grade. In them he found 429 different words. When only those words were counted that appeared more than five times or in not less than three books, the total of new words was reduced to 327. Does not the vocabulary of arithmetic books merit the critical analysis now made of all primary readers? To neglect the language aspect of arithmetic is to court failure.

Labeling and crutches. *Excessive labeling.* In many classrooms common practice requires children to label all numbers in the course of solving a problem. Since every quantity that represents something concrete must be tagged, considerable writing is done before the answer is obtained. Many teachers oppose full forms, and therefore teach various abbreviations, still persisting, however, in labeling every quantity thus: *h.m.* stands for *how many; l,* for *loss; g,* for *gain;* and *m.p.,* for *marked price.* This labeling is justified by a variety of reasons: it gives evidence of sustained thinking, teaches systematic arrangement of work, simplifies inspection by the teacher, and encourages accuracy.

Upon closer analysis we may doubt the effectiveness of the habit of labeling all numbers. It surely is not the method pursued in real life; one thinks about the problem and decides that if *a* were added to *c,* and the result divided by *b,* the number of "bales required" would be found. Only the answer is labeled. The more interested one becomes in the problem, the more impatient is he to reach the end of the calculations and to obtain the answer. It is not uncommon to find children who have obtained the correct answer going over the problem and inserting names to accompany the numbers. By doing this they avoid incurring the displeasure of the teacher. Nothing is gained by forcing the child to arrest the solution in order to associate every number with the concrete object that it represents. Labeling all numbers is often an obstructive practice.

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[4] Olive Gray, "Teaching Pupils to Read Arithmetic and Other Subject-Matter," *Elementary School Journal,* Vol. 26, April, 1926, pp. 607–18.

Clara M. Partridge, "Number Needs in Children's Reading Activities," *Elementary School Journal,* Vol. 26, January, 1926, pp. 357–66.

[5] S. S. Brooks, "A Study of the Technical and Semi-technical Vocabularies in Arithmetic," *Educational Research Bulletin,* Vol. 5, May 26, 1926, pp. 219–22. Ohio State University.

"Crutches." We frequently find auxiliary figures and markings taught children when new topics or operations are introduced. To teach a child, in column addition, "to put down the units under units, write tens in the corner, and then carry the figure to the other tens"; in subtraction, "to change the figures in the minuend"; in division, "to make checks under each figure of the dividend as it is brought down"—these are illustrations of forcing children to resort to "crutches." Number work with the aid of crutches takes on the following appearance:

$$\begin{array}{rl} 346 & \\ 218 & \\ 347 & \\ 236 & 3\cancel{2} \\ \underline{195} & 2\cancel{4} \\ 42 & \end{array} \qquad\qquad \begin{array}{r} {\scriptstyle 7\;10} \\ 3\cancel{8}4 \\ \underline{146} \\ 238 \end{array} \qquad\qquad \begin{array}{r} 34 \\ 14\overline{)4836} \\ 42^{\times} \\ \hline 63 \\ \underline{56} \\ 7 \end{array} \text{ etc.}$$

The tendency among recent writers is to reject all these crutches for reasons that are serious. It is argued that children who learn to use these aids develop habits of arrangement that must ultimately be discarded. Every auxiliary check tends to keep the process, already understood, a rational one, and retards rate of work. Crutches may therefore be devices that render temporary aid at an excessive ultimate cost. For these reasons the prevailing attitude towards crutches is distinctly hostile.

This is illustrated in the following use of crutch figures in the minuend:

$$\begin{array}{rl} \left.\begin{array}{r} {\scriptstyle 7\,10} \\ 9\cancel{8}4 \end{array}\right\} & 14 \\ \underline{365} & \\ 619 & \end{array}$$

Theoretical analysis may justify this position, but practical experience often refutes such a sweeping assertion. Teachers who grade their work carefully and give sufficient drill for each step find that many children can be brought to a desired level of proficiency without the use of accessory algorisms. But there are young pupils who, for one reason or another, cannot master a new solution without the aid of crutches. To force such children to abstain from the use of these aids reduces their speed and accuracy, and makes the learning process an experience of continuous worry. Such children should be encouraged, but not forced, to work without them. Before a final decision is reached in the matter of crutches, supervisors and teachers ought to consider these questions: May crutches be useful to slow and worrisome children? Granted that the crutch is not needed in the final product, may it not play a useful part in the learning stage? Have not most of us discarded crutches we found extremely useful as children? Is not the time required to write a crutch insignificant, and may not the worry about "not forgetting" cause greater loss of time? Have we any objective basis for the positive condemnation of all crutches? The entire question must be studied experimentally before a final decision is reached.

While we are awaiting accurate data in terms of which we may formulate answers to the questions listed above, it might be a helpful compromise to eliminate all temporary crutches and in their place give the child necessary warning in the early exercises. To illustrate:

Perform the multiplications indicated and *remember to carry and add after the first multiplication."*

The work would then be carried out as follows:

58	Child thinks 4 × 8, 32; says 2; writes 2.
4	Child thinks 4 × 5, 20, plus 3; says 23.
232	Writes 23.

This makes unnecessary the crutch so frequently seen in the following form:

$$\begin{array}{r} 58 \\ \underline{{}^{(3)}4} \\ 232 \end{array}$$

The correctness of all algorisms. The term *algorism* refers to the arrangement of an arithmetical solution. Arithmetic must be made the instrument of developing precision of speech as well as accuracy of thought in quantitative matters. Textbook writers object to the formula 5 ft.×4 ft.=20 sq. ft. on the ground that the multiplier cannot be concrete, and even if 5 feet could be multiplied by 4 feet, the conversion into square feet would be impossible. They suggest the form 5×4×1 sq. ft.=20 sq. ft., an algorism which the child hardly understands, and which makes the multiplier no less concrete to him. Nor is it correct to permit a statement like the following:

$$\begin{aligned} \tfrac{2}{5} &= \$3.60 \\ \tfrac{1}{5} &= \$1.80 \\ \tfrac{5}{5} &= 5 \times \$1.80 = \$9.00 \end{aligned}$$

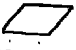
The correct form is, of course, $\frac{2}{5}$ of the money saved is $3.60. Children can be led to realize the absurdity of the incorrect statements by reference to concrete things.

No aim in arithmetic can be attained unless it is made a goal in all phases of the work. To insist on accuracy in number manipulations and to indulge in inaccurate statements will inevitably develop looseness of thinking in all arithmetic. On the other hand, we must not become too pedantic in matters of form. To be sure, the symbol, =, is not an equal's sign but a sign of equality, but such a distinction is hardly worth maintaining with young pupils. Unreasonable insistence on these matters, which must seem very small to children, harasses them. Teachers must be flexible in these matters and thus avoid fetishes.

The exaggerated importance of formal definitions. Arithmetic has a terminology of its own that children must learn. But ability to remember set definitions of important terms gives no assurance that they are understood. The child who glibly recites a formal definition of number, unit, or ratio is repeating a statement imposed by his teacher and formulated by a person possessing mature insight into number relations. Nor are all acceptable definitions correct and immutable. The definition of a fraction most frequently given by textbooks is the traditional "one or more of the equal parts of *a* unit." But ¾ is not always ¾ of 1. The traditional definition of a fraction does not suggest, even remotely, its utter incompleteness. Formal definitions are not the necessary aids which an older pedagogy assumed them to be.

But it must not be assumed that all definitions may be eliminated in the teaching of arithmetic. We must draw a distinction between very useful and relatively less useful definitions. It is obvious that the primary grades must be free from formal definitions because the children are too young to formulate the definitions themselves, and too immature to read meaning into those that are given to them.

Some terms merely help to identify arithmetical elements. They contribute nothing more helpful to a pupil. To define these with any degree of precision is unnecessary. The terms *addend, multiplicand, multiplier,* and *quotient* are examples. Recognition of these and ability to give illustrations of them meet very adequately all the demands of arithmetic.

But there are other terms that must be known more intimately. Definitions of *factor, rectangle,* and *denominator* do more than identify a class of arithmetical elements. They sum up essential characteristics that have a bearing on actual computations. The child who neither knows nor understands the definition of *rectangle* may fail to recognize the square and the oblong as rectangles or he may assign the parallelogram,  to the class of rectangles. Unless the term *altitude* is clearly defined and thoroughly understood the child frequently computes the area of a triangle by multiplying the base by one of the inclined sides; when he can find no horizontal side of the triangle, which he calls base, he is perplexed indeed. Here we have illustrations of definitions that enter into the very techniques of arithmetic.

Suggestions for teaching definitions. Experience indicates that certain teaching practices increase the value of learning definitions. These we shall now list.

The definition should come *after* the lesson has been developed, and after the *functions and characteristics* of the term in question *have been thoroughly grasped* by the child. First, children draw rectangles of different sizes and with different ratios between length and width. They then note, under guidance, that each figure has four sides and four right angles. Finally they are told that these figures are called *rectangles* and are asked to help formulate a definition of the term.

The definition must be *formulated by the class,* various children making improving contributions to the final statement. However inadequate the language used by the children may be, it is *they who must suggest the thought elements* in the definition, for only as they succeed in doing this are they giving evidence of comprehension.

After the component ideas of the generalization have been contributed by the pupils, the teacher may improve the language.

No fixed statement need be memorized by the children. *Learning the component ideas* in a definition is usually adequate. When called upon to give a definition of a rectangle, children should think of "four sides" and "four right angles" and should then express these two ideas in the best sentence they can formulate. A stereotyped definition is rarely, if ever, necessary or even desirable.

It will be found helpful to train children to give an original example before stating a definition. The child called upon to give the definition of *factor* or *rectangle,* says for the first, "5 is a factor of 10 or 15—a factor is a number that divides another number without a remainder"; and for the second, "The window-pane is a rectangle—a rectangle is a figure having four sides and four right angles." Children are often able to recite correct definitions and yet lack understanding. They are unable, therefore, to apply these definitions. The original illustration which precedes the definition gives ample evidence of adequate comprehension.

Definitions are summaries of experience. As experiences increase in number and variety the first definition loses its inadequacies and crudeness. Always, experience must run far ahead of definition and classification.

Problems for Study and Discussion

1. Select one of the standardized word lists, for example, that prepared by Thorndike or Gates. Examine a commonly used arithmetic textbook and check the words used in the problems of five different pages. Do you find many of the words in the first, second or last 500 of the list. What changes seem imperative?

2. Read about 50 examples and 50 problems in the book referred to above. What difficulties are introduced by the language of the exercises and the problems?

3. Prepare two tests on the following words: *gross, net, product, surface, area, profit, per, gain, reduce.* In the first test, ask the children to state in their own words the meaning of each word and to give an original illustration. In the second test, give for each word four meanings or four illustrations, of which only one is correct. What language deficiencies are revealed? How do the results of the tests compare?

4. Labeling all numbers in a problem is often an obstructing process. Why? Give specific instances.

5. List questions concerning the use of crutches that must wait for sufficient experimentation before final answers can be given. Plan an investigation for one of the questions on your list.

6. Turn the pages of a textbook in arithmetic and select (*a*) six terms that should be defined because they embody essential elements; (*b*) six that should not be defined but which children should be able to identify and illustrate. Justify the inclusion of each word in the two lists.

7. A definition must always remain a functional factor in a pupil's knowledge. What is meant by *functional factor*? What teaching processes tend to sharpen the *functional* aspect of a term or definition?

8. Plan an exposition, with appropriate references to teaching, of the following statement: Experience must always precede definition and classification. Do not limit the illustrations to arithmetic.

CHAPTER VI

MOTIVATION IN ARITHMETIC

Lesson aim and motivation. *Aim and motive differentiated.* As early in each lesson as possible pupils must become acquainted with the specific objective they are to reach. Proper mind-set is thus established. Each pupil can gauge the success of the lesson by ascertaining whether he personally has attained the aim. Knowing the goal helps pupils to direct their thinking towards a common end and to avoid many irrelevancies.

Before beginning the explanation of a short cut in multiplication, the teacher writes on the blackboard, 576×16⅔, and invites the class to a contest. While the pupils are still copying the numbers the teacher writes the answer on her pad rather ostentatiously and waits for the class to multiply 576 by ⅔, then by 6 and 1, and finally to add the three partial products. Again the class is challenged but many express the opinion that the contest is too one-sided. "What enables me to write the correct answer so rapidly?" asks the teacher. "You know a number trick," comes the reply. "What shall we learn in to-day's lesson?" brought as an answer, "The quick way of multiplying." Every pupil in the class thus knew the goal of the lesson.

The aim of a lesson should do more than foreshadow the goal. Properly used, it gives the child a simple and effective measure of his progress. At the end of each lesson the aim should be recalled and the children asked to judge whether they had attained their objective. We must remember that the affirmative answer cannot always be given truthfully. Further inquiry into the cause of failure—partial or complete failure—is often very instructive, and may yield a situation far bigger than the lesson itself. In such analysis one develops a critical attitude, which is most helpful in acquiring power to think.

The aim of a lesson refers to its immediate goal; the motivation seeks to justify the lesson to the pupils. An effective safeguard against teaching what is socially useless in arithmetic is to reveal to the pupil the value of the specific subject-matter to be learned. Not we, but the child, must be convinced of the need or the value of what is to be taught. Were teachers and textbook writers to motivate all they try to teach children, much which they now regard as necessary would undoubtedly be eliminated. When genuine motivation is established we may be reasonably certain that the class gives its attention willingly, that its interest is dynamic, that its self-activity flows freely toward a definite end, and that its coöperation is reasonably well sustained throughout the lesson.

How to motivate arithmetic lessons. The variety of methods of motivation in arithmetic is rich and easily applied to all grades. To these we now turn.

1. We establish motivation by showing children their need *of a process or set of facts.* In teaching business forms—bills, receipts, checks, single-entry bookkeeping—the topics are so intimately related to the child's experience that the mere mention of the aim of any of these lessons will serve as a motive. A first lesson on multiplication by fractions can be motivated by making one item of a bill, "1⅛ yd. at $2.40," and leading the pupils to foreshadow the aim of the lesson—"to learn how to multiply by fractions." Similarly, a lesson on division by a decimal can be motivated by giving the children a practical problem, the solution of which is found by dividing by 6. The problem is then changed in one respect only—the 6 is turned into .6. The children do not proceed very far with the second divisor before they inform the teacher that they do not know how to manipulate the decimal point. The pupils are then asked, "What must I teach you now?" and

the children state the aim of the lesson. These are typical illustrations of very many similar situations in arithmetic in which motivation can be achieved by showing that the new process is necessary in modern life.

2. A second means of arousing rational motive is *to relate new experience to an object or an end in which the child is directly interested.* A pupil who dislikes arithmetic but is eager for manual and vocational experiences may be attracted to arithmetic if its problems are derived from industrial activities. In making out the price tag for an article of furniture just completed the child must add a per cent profit to the cost of material and labor. Here we have the motive for teaching this child the initial lesson in percentage. In similar manner a child's interest in athletics may be utilized as a motive for arithmetical purposes. Four schools in athletic competition have thus far the following records:

School I	Won 4	Lost 3
School II	" 5	" 2
School III	" 5	" 3
School IV	" 4	" 1

These figures can be made the basis of an interesting series of oral arithmetic examples, and can be used to motivate the teaching of Case II in percentage by asking, "How shall we compute the per cent of games won by each?" Similarly, interest in physics, in geography, or in cooking can be made the means of stirring motive and arousing a desire for the mastery of certain arithmetical experiences.

3. A simpler means of attaining the same end, motivation, is *to appeal to a deep-seated urge or craving.* Because of the child's inherent love for play, early number work is made pleasurable through suitable games. Many of the mechanical processes of arithmetic take on a pleasing aspect if teachers appeal to the puzzle instinct, the instinct of curiosity, or the urge to overcome obstacles within the children's capacities. Reducing fractions to lowest terms and discovering least common multiples and greatest common factors are not dull pro-

cesses when these are presented as number games that challenge ingenuity. Even adults spend many hours in puzzle-solving. When all other attempts at motivation fail an appeal to an instinctive craving will often prove effective.

4. *The desire for growth, for self-improvement,* is potent in every normal human being. This becomes a motivating force in many forms of learning. The competitive spirit is easily aroused. But instead of directing it towards others we try to turn it inward. Let the child try to compete with himself, to do better than he did yesterday. In classes where children graph their individual weekly records there is unmistakable desire on the part of most of them to keep their curve of progress from "turning down." They strive for the upward turn of the graph and find solace in the plateau. Work that is unusually meritorious for a particular child should be praised or exhibited. Occasionally it should be sent home with a written comment by the teacher. Such recognition is undoubtedly a spring that releases human effort.

Observations concerning the limitations of motivation. A note of caution is sounded by many critical students who point out that too frequently the motivating situation is sought in adult life—not in that of the child. Are we making arithmetic natural and necessary by arranging facilities for playing store, playing number games, or for playing circus because each of these situations requires number manipulations—counting, adding, making change, multiplying, and what not? Is motivation genuine because the number questions are centered around, "Going to the Movies," "A Trip Around the City"? Are not these experiences with their simulated enthusiasm adult made? To these situations the dull child responds for the moment, but is left cold when actual arithmetical examples must be solved. The bright child resents the all too obvious sham. Why not proceed on the safe assumption, the critic asks, that normal children are intellectually alert, and eager for new experiences? Does not our numerical system motivate itself because of its rhythmic repetitions? Do not children recognize the practical need for number? Under normal conditions children strive to know more and to be more skillful. Will not the normal reactions of the child make elaborately planned motivation unnecessary? These apprehensions, implied and expressed, are justified when we teach subject-matter that is inherently interesting. But each new unit of graded subject-matter, though vitally necessary for the development of the entire subject, may not relate itself to the child's immediate needs. Ultimate need is no motivation for a growing, active child. It is argued that motivation, even when it is slightly artificial, is occasionally necessary.

At times we may not be able to motivate a given lesson in arithmetic. Under these conditions the lesson proceeds without motivation. Often the motivation comes at the completion of the act of learning. With the development of the desired degree of skill in using fractional equivalents of percentage forms, the child sees the superiority of the multiplier ⅜ over .37½ when 37½% of 6.568 must be found.

To motivate does not always mean to make agreeable. Motivation may make a task seem good fun, but it must always make it seem reasonable or necessary. Once the child sees the need of drill to insure mastery, the drill is motivated. To be sure, he may not enjoy the round of repetitions, but he now regards the task as useful, not as an arbitrary imposition by an imperious teacher.

Success and failure as motivating agents. Thorndike challenges the statement that thinking is initiated by a difficulty or a failure. He argues that these circumstances may inhibit rather than stimulate thinking.[1] For illustration he offers, "It is hard for me to add three eight-place numbers at a glance. My existing sensory-motor connections are inadequate to playing a golf course in 65." These failures either inhibit Thorndike, or set up annoyances for him. He therefore believes it unwise to teach cancellation by showing the child that multiplying all the numerators and all the denominators and then reducing the result to lowest terms is far more time consuming than it need be.

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[1] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), p. 280.

"To emphasize before teaching cancellation, the inadequacy of the old full multiplying and dividing would, moreover, not only be uneconomical as a means of teaching cancellation; it would amount to casting needless slurs on valuable past acquisitions, and it would, scientifically, be false. For until the pupil has learned to cancel, the old full multiplying is not inadequate. It is admirable in every respect. The issue of its inadequacy does not fully appear until the new method is found. It is the best way until a better way is found."

All of this is true and yet it does not necessarily contradict teaching procedure that is commonly approved. A difficulty may discourage and set up inhibitions to be sure, but ready success may make a process or an experience seem so familiar that it holds nothing more for us. Neither success nor failure, alone, is sufficient to stir dynamic response. Failure plus something else—difficulty plus something else—are needed. The something else may be a desire to do better, to do as well as others, or to have something that is considered useful. If mere failure were adequate stimulation, then those who fail most—the least intelligent—would be the most stimulated. The least intelligent fail most because they lack the penetration to interpret their failures and their inadequacies.

To show an intellectually alert child that his process of doing all the multiplications first is time-consuming may motivate the cancellation process for him. Of course he cannot hope to save time and energy until he has mastered the process, but, for that child, the desire to work in the most economical and most approved fashion quickens the learning process.

Failure uninterpreted, failure whose implications are not understood, failure in situations that transcend our ability, will not serve as a motivating agent.

Success may accelerate learning more effectively than failure. The stimulating effect of success was apparent in the child who kept at her arithmetic for two hours because she felt, for the first time, that she really understood the functions of fractions and how to use common fractions in her computations. Success is accompanied by pleasure of attainment, and it generates a growing feeling of self-confidence. The child whose swimming improves continues his effort, and practices for longer periods. But when he reaches a long plateau in his learning curve he exhibits diminished interest and his practice becomes less frequent.

Neither success, *per se*, nor failure, *per se*, is necessarily a stimulation to further effort. The implications of success or failure may act as either accelerating or retarding factors in the learning process. The true psychology of success and failure still remains to be discovered.

PROBLEMS FOR STUDY AND DISCUSSION

1. List as many interests of children of school age as you can. Think of personal development, variety of experiences, relation to favorite things and people, vocational ambition, achieving distinction in the group, dramatic presentation, participation in activities, etc. Which of these can be utilized for motivating the learning of arithmetic?

2. Characterize, with supporting reasons, each of the following statements as generally true or generally false, or frequently true, or frequently false:

(*a*) Individual records of week-by-week progress stimulate the gifted, more than the mediocre children.

(*b*) An assignment that is difficult discourages the children unduly.

(*c*) Children who participate in setting up the objective of a lesson are the most actively interested in the lesson.

(*d*) There is a rich body of scientific data on motivation.

(*e*) The pupils who are most proficient in arithmetic have the most favorable attitude toward the subject.

(*f*) Because school children make little use of arithmetic, they do not respond when we motivate arithmetic lessons by revealing the practical applications of what we are to teach.

(*g*) Most children have a natural interest in number.

3. As a result of your study of this chapter, formulate a definition of *motivation* which you think the author would accept. Do you agree with the definition? Why?

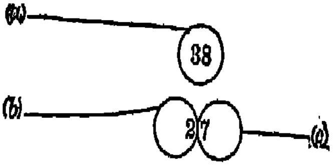
4. What topics in arithmetic should be taught without motivation? Justify the unmotivated teaching of each.

5. Devise ten questions, not more than five of which are to be memory questions, designed to test a fellow student's grasp of the content of this chapter. What is each question designed to reveal? Do you think it achieves its purpose?

CHAPTER VII

GRADATION IN ARITHMETIC

Gradation as a factor in successful learning. *Illustration of importance of gradation.* A teacher was trying to carry out her course-of-study assignment by teaching the addition of 38 and 27 in the prescribed form: 38, 58, 65. The children found great difficulty in keeping the sequence 38+20, 58; 58+7, 65. The teacher then numbered the successive steps thus:



That too failed. Being resourceful, she evolved a patriotic device: the circle around 38 was drawn in red; the circle around 20, in white; and the circle around 7, in blue. But nothing the teacher did reduced the difficulty.

Let us grant that it is useful to teach children to add two such two-place numbers without paper and pencil and that the method suggested, 38, 58, 65, is better than (30+20)+(7+8)=50+15=65. The two numbers 38 and 27 present almost all the difficulties such a number situation embodies. Why not the simplest step first, then the next graded step which contains one and only one new element? To teach this form of addition the teacher should set aside five minutes each day for a number of days. The following tabulations show (*a*) what is done each day and (*b*) the graded development of this form of addition:

The Gradation	*The Reasons*
First day. Give practice in the following: 3 + 2, 30 + 20; 4 + 3, 40 + 30; etc.	There is little here that the children do not know.
Second day. 30 + 20, 34 + 20; 40 + 30, 46 + 30	The combinations are almost the same except for the number of units in one number.
Third day. 34 + 20, 34 + 23; 46 + 30, 46 + 32 Form required: 34, 54, 57; 46, 76, 78	A second number of units is introduced but the combined number of units is less than 10.
Fourth day. 34 + 20, 34 + 26; 46 + 30, 46 + 34 Form required: 34, 54, 60; 46, 76, 80	Now the two units places yield 10 but no more.
Fifth day. 34 + 20, 34 + 27; 46 + 30, 46 + 39 Form required: 34, 54, 61; 46, 76, 85	The difficulty of bridging the 10's is here introduced.

It is now apparent that the last step in a chain of progressive difficulties was made the first step in teaching. Small wonder that the children seemed inextricably confused in trying to learn this form of addition.

Another illustration may convince the reader that proper gradation in such subjects as elementary mathematics, grammar, and science is of utmost importance; that it is probably the most important single agent in eliminating the difficulties which these subjects hold for children. In learning how to subtract common fractions the class came to the type in which the fraction of the subtrahend is larger than the fraction of the minuend. The teacher unwisely began the lesson with $5\frac{1}{4}-2\frac{3}{8}$. Clearly the children had to perform six important major steps: (*a*) to reduce both fractions to a common denominator; (*b*) to recognize that $\frac{3}{8}$ cannot be taken from $\frac{1}{8}$; (*c*) to secure $\frac{8}{8}$ by changing one of the units of the whole number 5; (*d*) to fuse the newly acquired $\frac{8}{8}$ and the $\frac{2}{8}$ of the minuend and secure $\frac{10}{8}$; (*e*) to subtract $\frac{3}{8}$ from $\frac{10}{8}$ and secure $\frac{7}{8}$; (*f*) to subtract 2 from 4, not 5. Of these six steps (*b*), (*c*), (*d*), and (*f*) are altogether new. Are we justified in permitting four new procedures to arise at once and thus make arithmetic seem unduly difficult? Would the following not have been a more helpful gradation and have led to less discouragement?

Step 1. $5-2$; $5-\frac{3}{8}$; $5-2\frac{3}{8}$
Step 2. $5-2$; $5-2\frac{3}{8}$; $5\frac{1}{8}-2\frac{3}{8}$
Step 3. $5\frac{1}{8}-2\frac{3}{8}$; $5\frac{1}{4}-2\frac{3}{8}$

We assume, of course, that the children have learned to solve the following types: $\frac{5}{8}-\frac{3}{8}$; $2\frac{5}{8}-\frac{3}{8}$; $\frac{8}{9}-\frac{1}{3}$; $7\frac{8}{9}-2\frac{1}{3}$. The borrowing difficulty, properly graded, is no great worry; imperfectly graded, it becomes a mountain of small trials.

Psychological implications of gradation. The psychological implications of gradation are significant. The specific character of an arithmetic skill has already been mentioned and will be elaborated in our diagnostic studies. An enumeration of some of the specific skills that make up skill in addition includes the following: mastery of number combinations; ability to bridge tens, 5+7, 15+7; ability to carry a combination to higher decades, 15+7, 55+7, 75+7; ability to write addends correctly; habit of checking the result. Skill then consists of a complex of constituent skills, each of which is most specific in character. The specificity of skill, therefore, adds new importance to gradation, for it teaches that we must develop general skill by developing single skill elements in progressive sequence. This is precisely what was suggested in the illustrations above.

Learning is essentially the acquisition of a hierarchy of habits. Stone[1] believes that before children are asked to add six three-place numbers, they should have acquired the preliminary graduated set of habits tabulated on page 94.

The term *apperception,* to-day in ill repute, refers to a very useful concept: we acquire new ideas by interpreting them in terms of related ideas previously held. In proper gradation we supply each new step with a set of suitable inter-

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[1] John C. Stone, *The Stone Arithmetic, Teachers' Manual, Grades 1 to 6* (Benj. H. Sanborn & Co., 1928), p. 36.

Order of Step	A	B	C	D	E
The Step	3 5	4 2 3	21 34	38 26	17 2
The Reason	Merely an item out of the 100 additive combinations.	Same as A with 3 addends.	Application of A and B to 2-place numbers.	First carrying.	Incomplete form of C and D.

Order of Step	F	G	H	I	J	K
The Step	8 6 5	17 5	8 9 5	7 6 9 3	5 7 6 8 4	365 426 978 342 685 376
The Reason	Total of 3 addends is above 10; merely one step beyond E.	Step E applied to a case whose units are more than 9.	Same as F and G, sum above 20.	F carried one step higher.	I carried higher.	The goal: The combination of all previous steps.

preting ideas and skills. In the addition of two two-place numbers, the work set for any one day becomes the interpreting experience for the processes of each succeeding day. Similarly, in the illustration of subtraction of fractions, step 1 is the apperceptive basis for step 2, and steps 1 and 2 become an apperceptive basis for step 3. In all situations concerned primarily with the development of skill proper gradation is of extreme importance in facilitating the learning process.

Proper gradation inspires confidence in one's ability. Although a child may not be able to master at once a complicated skill, he finds he can learn the progressive specific skills. Learning thus becomes more pleasant, and associated more often with success than with failure. Under the guidance of a gifted teacher, therefore, learning becomes a series of adventures in success.

How detailed is effective gradation? The teacher who is intellectually sympathetic takes the attitude of the immature pupil and discovers, one by one, the successive difficulties that present themselves in the unfolding of the subject. Such a teacher introduces into a lesson a gradation that goes far to remove obstacles from the learner's path.

Let us assume that division by a decimal has been explained carefully by the teacher and understood clearly by the class. It does not follow that because the children can solve examples used by the teacher in the course of instruction in which the divisor is .6 or .8 or .15, that they can therefore solve any case that comes under the caption, Division by Decimals. When the example takes the form of $.15\overline{)2.455}$ or $.12\overline{)36.48}$ no difficulty is experienced, but when a slight change is made so that the example becomes $.15\overline{)245.5}$ or $.15\overline{)2455.}$, many are puzzled, for there is an apparent insufficiency of decimal places in the dividend. The writer has seen children able to solve $.15\overline{)65.75}$ but unable to obtain the correct answer to $.15\overline{)657.5}$; able to solve $.15\overline{)657.5}$ but unable to manipulate the necessary elements in $.15\overline{)6575}$; when questioned as to the cause of the difficulty the children replied, "There is no decimal point in the dividend." The reader might assume that the instruction was inefficient and that the children were taught that to solve $.15\overline{)65.75}$, "we move the decimal point two places in the divisor and two places in the dividend." Such an assumption would be unwarranted, for when the pupils were asked to explain the division of 65.75 by .15 they said, "We multiply the divisor by 100 and then do the same to the dividend." When they were requested to apply the law to the division of 65 by .15 they moved the decimal point in .15 two places to the right, but since there was no decimal point in the dividend, they multiplied by 100 by adding two ciphers and proceeded as follows:

$$\begin{array}{l} \quad\; 4\;33\tfrac{5}{15} = 433\tfrac{1}{3}. \\ 15.\overline{)65\;00} \\ \quad\;\; \underline{60} \\ \quad\;\;\; 50 \\ \quad\;\;\; \underline{45} \\ \quad\;\;\;\; 50 \\ \quad\;\;\;\; \underline{45} \\ \quad\;\;\;\;\; 5 \end{array}$$

But without this added direction they seemed unable to proceed. In the teaching of division by decimals careful gradation demands the introduction of the following steps, each taught only when the preceding one is mastered.

Step	*Reason*
1. $5\overline{)54.25}$	1. This is a known procedure.
2. $.5\overline{)54.25}$	2. This introduces the problem of division by a decimal and shows the difference between the known form and the new.
3. Divide by .6, .7, .9, $.6\overline{)37.86}$, $.7\overline{)54.67}$	3. Drill in dividing by decimals of one place.
4. Divide by .12, .14, .15, $.12\overline{)38.64}$, $.14\overline{).322}$	4. Applies the same law to decimals of two places
5. Divide by 1.2, 1.4, 1.14, $1.2\overline{)38.64}$, $1.4\overline{).322}$, $11.4\overline{)7.98}$	5. Introduces the idea that the whole number in the divisor has no influence on the mode of procedure.

Step	*Reason*
6. Divide 32.2 by .14	6. The number of places in the dividend is insufficient and must be made up by the addition of ciphers.
7. Divide 322 by .14	7. Dividend a whole number.
8. (*a*) Increase the decimal places to 3 places	8. (*a*) Carrying out same law with examples a little more difficult.
(*b*) Divisor, a mixed number with 3 decimal places, for example, 1.215	(*b*) Showing that whole number, again, has no influence on mode of procedure.
(*c*) Divisor, 3 decimal places; dividend less than 3 decimal places	(*c*) See Steps 6 and 7.
9. To divide by 240, 2400, 800, etc. (*a*) $2400\overline{)5468.76}$ (*b*) $200\overline{)5.468}$ (*c*) $200\overline{).5468}$	9. While this is not dividing by a decimal, a related law operates and helps to simplify the process, for example, $24.00_{\times}\overline{)87.65_{\times}42}$. (*a*), (*b*), and (*c*). The gradation here indicated is explained above when similar gradation is shown in the division by a decimal, Steps 6 and 7.

In planning the application of a new principle in arithmetic the teacher must list all useful forms or types, and then so grade these that as the round of applications is met, only one new difficulty at a time presents itself to the child.

Problems for Study and Discussion

1. Select three important arithmetic processes, and indicate for each the successive steps in a well graded organization. Be sure to justify each step.

2. Illustrate the following statement by reference to one of the processes in your answer to the first question: Learning is essentially the acquisition of a hierarchy of habits.

3. "Learning is a process of establishing a series of related specific bonds." Does this statement lend greater or less importance to fine gradation in teaching arithmetic? Why?

4. Is there a denial of transfer of training in emphasizing gradation in teaching arithmetic? Why?

5. Examine any textbook in arithmetic in current use. Find illustrations, if possible, of the following:

(*a*) Effective gradation
(*b*) Inadequate gradation
(*c*) Over-fine gradation.

In each instance, justify your judgment of the gradation, and indicate the changes, if any, that you would make.

CHAPTER VIII

HABITUATION IN ARITHMETIC

Arithmetic a subject of habits. School activities may be grouped into studies relatively richer in content and studies relatively richer in form. History, descriptive geography, literature, and nature study are illustrations of the first; arithmetic, grammar, locational geography, and map study come under the second. The first group of studies is rich in human relations, the second group sums up settled facts and fixed associations. We must recognize that this is not a fixed classification, for form and content cannot be divorced completely. The controlling purpose of all teaching activities is to reduce necessary associations to habit. Because mastery of arithmetic requires the acquisition of so many habits, the drill lesson plays no small part in it.

What habits should arithmetic develop? The habits that must be developed through the mastery of arithmetic may be grouped under three heads. There are, first, the *habits of subject-matter*. Every item in the multiplication tables, the additive combinations, the facts in the common tables of weights, all the modes of computation, notation of decimals, short cuts, per cent equivalents of common fractions, modes of checking answers—these constitute but a small part of the subject-matter which must be so thoroughly mastered that they in turn master the child.

The second set are the *habits of procedure*. Throughout the teaching of arithmetic teachers must insist that the pupils (1) adhere to a systematic arrangement of all written solutions; (2) use accepted forms of business papers; (3) scrutinize an answer and check it before going to the next problem; (4) examine a problem before solving it and decide what is given and what is to be found; (5) label necessary items in the solution of an example; (6) reduce the numbers in difficult problems to much smaller ones and then attempt to evolve a solution; (7) translate new problems into diagrammatic form and seek a solution; (8) write all figures neatly and legibly. These procedures are constantly emphasized by the teacher in the hope that by dint of repetition they will sink deep and will be performed through force of habit.

Habits of mental behavior constitute the third group. Throughout the teaching of arithmetic we strive to develop a sense of accuracy, increased concentration, power to imagine new relations, power to reason and judge. It is hoped that as a result of years of study of arithmetic children will acquire a mode of attacking a problem that involves quantitative relations.

These three sets of habits mark the permanent results of the subject, the final ends which the teacher seeks to attain. Scientific standards in arithmetic for determining progress and capabilities of pupils measure the habits which have been inculcated in children.

Functional and formal drill. *The two forms of drill contrasted.* What teacher cannot match the story of Willie, who rushed into the classroom and in great excitement informed his classmates, "My big brother has came back from the Navy." "Has came?" the surprised teacher asked, and Willie replied, "Yes, he has just came back." "Give me the principal parts of the verb, to come," asked the teacher. To which Willie promptly replied, "come, came, coming, come."

Unless a fact is learned in its natural association it retains its isolation. The verb forms have no influence on Willie's speech because he knew them out of context only. Parallel instances can be cited from arithmetic. A child who can give the correct answers to $5+7=?$, $5\times7=?$, $5\times?=35$, $?\times7=35$, $?+7=12$, may nevertheless be unable to solve such simple problems as the following:

I had 12 cents and lost 7; how many did I have left?

If each of 7 boys has 5 marbles, how many marbles have they all?

Children need 35 marbles in a game that 5 are to play; how many should each child bring?

To know a fact in isolation is one thing; to make it function in a real situation, quite another. Practice on mere form 7+5=?, 5+?=12, 5×?=35 is formal drill; practice in solving a large number of problems like those cited above is functional drill. Although there is considerable loss in carrying over mastery developed in formal drill to functional situations, a certain amount of formal drill is inevitable. It is important, therefore, that provisions be made for both forms of drill so that children will gain facility in making appropriate applications of their knowledge. To argue that formal drill is more or less important than functional drill is idle because the two types are mutually supplementary, one insuring the correct form, the other the appropriate application.

As a rule, practice in decimal notation is reduced to reading and writing meaningless numbers. To give such practice a functional quality, problems like the following should be devised for solution by the pupils.

Dates	Rainfall
The Week of	Cu. in. per sq. in. of area
July 5–11	.450
12–18	.208
19–25	.150
26–Aug. 1	1.050
Aug. 2– 8	1.045
9–15	.790
16–22	.815
23–29	.515
Total	

Explanation: The amount of rain that falls is measured in cubic inches on each square inch of land. The smaller the number, the less the rainfall. Read the rainfall for each week as presented in the table.

Questions: 1. Name the weeks when the average rainfall was about .5.

2. What weeks made the dry spell of that summer?

3. What was the average rainfall of each month? of both months?

4. What two weeks average about .8?

5. In what week did twice as much rain fall as in the preceding week?

6. In what week did three times as much rain fall as in the third week?

The cow, Star Elsie, gave 1742 pounds of milk in January. In each pound of milk there was .0461 of a pound of butter fat. Read each of the following monthly records:

Month	Pounds of Milk	Butter Fat per Pound of Milk	Total Number of Pounds of Butter Fat
January	1,742	.0461	
February	1,690	.0485	
March	1,574	.0504	
April	1,226	.0490	
May	1,202	.0466	
June	1,251	.0481	

Questions *

1. In what month did Star Elsie give least milk? most milk? What was her monthly average?

2. In what month was her milk richest in butter fat? poorest? What was the actual difference between these months?

3. Fill out the items of Column III of this table.

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* Based on suggestion in Thorndike's *Psychology of Arithmetic*, p. 273.

Functional drill determined by need as well as by use. Our illustrations tended to show that repeated use in a natural situation makes drill functional. Practice that is undertaken to meet a definite personal need may also be regarded as functional drill. The effectiveness of a drill lesson is determined directly by its capacity to meet a personal need or overcome a specific weakness. The drills that are best, therefore, are those following diagnostic tests that reveal individual or group needs. Too often one observes a drill lesson in which equal emphasis is laid on every constituent item by every pupil. Clearly this is most unwise, for the weaknesses of all pupils are not identical, nor are the items of the lesson of equal difficulty. An analysis of practice material offered in standard books reveals the following disconcerting facts:[1] there is four times as much practice on 2÷2 as on 8+8; eight times as much on 2×2 as on 8×9; eleven times as much on 2−2 as on 17−8; and over forty times as much on 2÷2 as on 72÷8 and 72÷9 together.

Authors of textbooks provide drill material with no definite relationship to the needs of pupils. They guess, and their guesses are no adequate bases for these decisions. Why drill 154 times on 2+3 and only 61 times on 8+9? Is 2+3 more than twice as difficult as 8+9? Why drill 214 times on 2−1 and only 27 times on 17−8? Is 2−1 eight times as difficult as 17−8? Why provide 458 repetitions of 2×4 and 82 repetitions for 9×8? Is 2×4 over five times as difficult as 9×8?

Variety of drill. To maintain interest in drill, variety of practice must be provided. Adequate variety in drill may be introduced in at least three ways:

a. By changing the objects, materials, and activities. The changes from formal to functional drill, from oral to written practice, from abstract examples to concrete problems, from rapid questions and answers to a game, from a game to a serious communal activity like planning the purchases of Thanksgiving baskets—all these offer pleasant variety and help maintain interest.

There is an important limitation in employing games and communal activities as modes of drill in a subject as formal as arithmetic. Is interest in the number game focused on the competition, on the game, or on the combination of elements which makes the number? In playing store a particular child makes *a* purchase. In the whole activity the child has performed one arithmetical operation. What does this child gain from the remainder of the "playing-store" activity? Does he actually perform all the arithmetical operations as he watches his classmates make their purchases? If the children are really playing store, they probably enjoy the playing but learn few number combinations. We do not argue for the general condemnation of the practice of introducing social activities into arithmetic, but rather for a clear realization that the arithmetic gains are relatively slight. There are, however, other outcomes from these social activities, which are at least as socially desirable as mastery of number facts.

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[1] E. L. Thorndike, "The Psychology of Drill in Arithmetic: The Amount of Practice," *Journal of Educational Psychology,* Vol. 12, April, 1921, pp. 183–194.

b. By changing the sense appeal. In simple combinations like 3+4 we may ask children to listen to pencil tappings or bell rings. Let each circle represent a sound made by the teacher.

The children listen and answer as promptly as possible.

ooo and how many more make 7?
oooo and how many make 7?
ooo and oooo make how many?
oo oo ooo less ooo make how many? etc.

Children are then called upon to hop or handclap or tap or bounce a ball the number of times necessary to answer the following:

3 and how many more make 7?
4 and 3 make what?
7 less 4 will leave?
A 3-cent book and a 4-cent book cost how much?
I had 7 cents and lost 3 cents. What did I have left?

Here we have number combinations learned through the kinæsthetic as well as through the auditory and the visual sense. Care must be exercised that tappings do not become a counting crutch. The aim must always be to reduce the combination to habit.

c. By changing the combinations. Eight children may be arranged in a row in the front of the room. Various classmates are called up to show how many 2's in 8; how many 4's in 8; how many 5 and 3 are; 6 and how many more make 8; and so on until an adequate number of repetitions is insured. The statement, 12 inches=1 foot, need not be repeated in this very form. Let the children look at it and then answer the following questions: How many inches in 2 feet? in ½ foot? in 1½ feet? in 2 feet and 1 inch? in 1 foot and an inch? How many feet in 24 inches? in 6 inches? in 18 inches? in 12 inches? These changes of statement rivet attention on what is to be habituated.

Drills to introduce a skill vs. drill to maintain a skill. A distinction is often made between drills that begin a new skill or a new habit and those that tend to maintain a skill already established. We shall presently consider, in detail, the first type—the drill that initiates a new skill by motivation and thorough understanding of what is to be habituated. At this point we stop for a passing illustration only. Let us assume that children must learn 4×9, 6×6, and 3×12. Instead of requiring children to repeat these combinations an endless number of times, it is often helpful to present the following on the board:

$$\boxed{\begin{array}{r} 4\times 9=36 \\ 6\times 6=36 \\ 3\times 12=36 \\ 9\times 4=36 \\ 12\times 3=36 \end{array}}$$

and then to assign the following for solution:

$$\begin{array}{ccccc} 14 & 16 & 13 & 19 & 112 \\ \underline{\times 9} & \underline{\times 6} & \underline{\times 12} & \underline{\times 4} & \underline{\times 3} \end{array}$$

The children are permitted to refer to the combinations on the blackboard. With each succeeding example the new combinations become more familiar. The more rigorous and abstract drill is thus motivated, and the functional aspect of the drill is assured.

The second type—drills to maintain the skill that has been acquired—may be both formal and functional, but they must be given in progressively spaced repetitions. Twice a week at first, once a week later, and even once a fortnight may suffice. Thus it is important to provide repetition of the per cent equivalents of common fractions or of column addition long after these facts and processes have been understood and have begun to function as habit. *Continued practice, with progressively spaced intervals,* tends to keep alive a newly acquired set of skills.

Isolated vs. mixed drills. In isolated drills all practice is focused on one process or on one unit skill; in mixed drills practice is given on related skills. An analysis of the exercises below will make clear the difference.

Mixed Drill	*Isolated Drill*
Perform the process indicated:	Find the sums of the following:
(1) $\frac{1}{2}+\frac{1}{3}$; $\frac{1}{4}+\frac{2}{3}$; $\frac{1}{6}+\frac{3}{5}$	(1) $\frac{1}{2}+\frac{1}{4}$; $\frac{3}{4}+\frac{3}{8}$
(2) $\frac{1}{2}-\frac{1}{3}$; $\frac{2}{3}-\frac{1}{4}$	(2) $\frac{1}{2}+\frac{1}{3}$; $\frac{3}{4}+\frac{5}{6}$
(3) $\frac{1}{2}\times\frac{1}{3}$; $\frac{2}{3}\times\frac{3}{2}$	(3) $\begin{array}{r}\frac{3}{8}\\ +\frac{3}{4}\\ \hline\end{array}\quad\begin{array}{r}\frac{5}{6}\\ +\frac{2}{3}\\ \hline\end{array}$
(4) $\frac{1}{2}\div\frac{1}{3}$; $\frac{2}{3}\div\frac{3}{2}$	(4) $1\frac{1}{4}+3\frac{1}{8}$; $5+\frac{7}{8}+5\frac{7}{8}+1\frac{1}{4}$

Evidence gathered shows clearly that mixed drills are more effective than isolated drills for maintaining a primary skill already established, that is, through the mixed drills children learn to work more accurately and more rapidly.[2]

The essential elements of effective drill. Since the final purpose of the drill lesson is to make certain knowledge and processes function as habit, the laws governing effective drill and the laws of habit inculcation must be identical. Any factor which quickens habit formation is equally effective in improving the drill lesson.

1. *Motivation.* Without the child's coöperation habit formation is retarded; with it, accelerated. It is necessary, therefore, to help the child understand the value of the habit he

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[2] A. C. Repp, "Mixed vs. Isolated Drill Organization," in the *Twenty-Fifth Yearbook* of the National Society for the Study of Education, Part II (Public School Publishing Co., 1930), Ch. vi.

is asked to acquire. Much of the tedium of a drill lesson is eliminated when the pupil regards the drill as necessary to attain a desired end. Adequate motivation can be assured in a number of ways. These are set forth in the discussion of motivation.

In a sixth-year class certain fractional equivalents of percentage forms were developed. A little rapid questioning evoked hesitant responses. When the teacher expressed surprise at the inability of the children to make rapid applications, they justified themselves, "Well, we haven't learned all these yet." They meant, of course, that they had not memorized, $\frac{5}{6}=83\frac{1}{3}\%$, $\frac{1}{6}=16\frac{2}{3}\%$, $\frac{7}{8}=87\frac{1}{2}\%$, and the others. "But you must know them as you know 5×5, if these new facts are to be of service to you," persisted the teacher. "What will you do?" The prompt reply was "Learn them." And learn them they did. We do not assert that properly motivated drills become pleasurable experiences to which children look forward with joyous anticipation. We do maintain that a motivated drill takes on the aspect of reasonableness characteristic of a socially necessary task. And that is sufficient. The teacher need not spend time and effort making useful things agreeable. She must rather concern herself with keeping what is necessary from assuming the aspect of an arbitrary task imposed by the school. It is vital that the child understand that life and not the teacher makes these demands on his time and his thought.

2. *Adequate comprehension.* No drill lesson should ever be given unless it has been preceded by a period of thorough exposition. No number combinations and no arithmetic process must ever be subjected to drill unless the teacher has reasonable assurance that the children understand the ideas and the relationships involved.

3. *Repetition.* The subject-matter that is understood and motivated may now be repeated in an effort to insure their functioning as habit. Mere repetition is too frequently credited with being a potent learning factor. "Practice makes perfect" is not always true. Motivation and comprehension are per-

haps more important than repetition. *The distribution of the repetitions is more significant than the number of repetitions.* To drill on a set of facts or skills everyday for a week and then to drop all drill lead to forgetting. Putting knowledge into cold storage does not keep it alive. It is important to provide for progressively distributed learning periods: a daily drill once a day for the first week; on alternate days during the second week; semi-weekly during the third week; once a week for the following month; once in two weeks for the remainder of the semester. Care must be taken however to keep the drill periods *very short.* Better results can usually be obtained in two drill periods of five minutes each than in one period of twelve or fifteen minutes. We have not yet determined experimentally exactly how long we should make drill periods for different grades and for different operations in arithmetic.

4. *Focused attention.* Unless attention is focused on the repetitions they become aimless practice that does not insure habituation. Children who repeat, "36 inches equal 1 yard," while giving themselves to a stream of pleasant imagery of games and associated experiences, are merely mouthing words and producing empty sounds. The equivalents, 36 inches and 1 yard, are not being associated and reduced to a permanent bond. Children who write tables as punishment perform mechanical penmanship without attending to the associations involved. Such drills deceive both children and teachers; they involve serious waste and engender a deleterious attitude towards school subjects.

5. *Conditions introduced to sustain attention.* To offset the monotony all too frequent in drill lessons, experienced teachers introduce those practices of class management that are designed to intensify attention. The questions should be (*a*) distributed as rapidly as circumstances permit, and (*b*) asked of the whole class. To call on a child and direct the question to him will slow up the lesson unduly and relieve the rest of the class of the responsibility of having an answer ready. For the latter reason it is also (*c*) unwise to call on

pupils in regular sequence, whether by alphabet or by seating arrangement. The first few called promptly eliminate themselves from the lesson and revert to memories good and sweet. Such a degree of comfort is usually evidence of ineffective conduct of a drill lesson. (*d*) Answers should be individual, not concert, responses. Collective answering is deceptive. Few think and many respond. (*e*) All the appropriate devices (explained earlier in the discussion) for securing variety of practice must be introduced if the drill lesson is to be maintained on a high level of effectiveness. Change in the form of statements and multiple sense appeal go far to insure necessary variations.[3]

Time standards in drill. Timed drills are valuable insofar as they prevent excessive slowness, on the one hand, and, on the other, set up a reasonable rate which discourages the inaccuracies accompanying the feeling of being driven. We must avoid such directions as the following:

a. "See how long it takes you to do this exercise." Here no standard is suggested and each pupil tends to set his own rate.

b. "Practice this exercise until you can do it in ten minutes." Are we not discouraging the pupil who can complete the task in eight minutes? Why encourage such pupils to develop a tempo slower than is normal for them?

c. "Practice this exercise until you can do it rapidly." What is meant by rapidly? The direction is entirely too vague to be helpful.

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[3] The reader may be interested in comparing this analysis of an effective drill lesson with the one offered by Knight in the study by Guy M. Wilson, "Arithmetic, Research in the Elementary School Curriculum," pp. 35-109, *Third Yearbook of the Department of Superintendence*, National Education Association, 1925.

Eight provisions are offered:

1. Drill should be on the entire process.
2. Drill should come frequently and in small amounts.
3. Each unit should be a mixed drill.
4. Drill should have time limits.
5. Drill should have accuracy standards.
6. Examples in a unit of drill should be in the order of difficulty.
7. Drill units should include verbal problems.
8. Drill should facilitate diagnosis.

d. "Do this exercise until you score 100%." Serious waste is involved in such drill because the child repeats all the items even though he errs in only a few. His interest is killed because he repeats what he knows; his attention is diffused instead of being focused on the items he really does not know.

It is better to set the time that a cumulative record of pupils' work would indicate as reasonable. Then we may ask the child to note his time in order to ascertain how much better he can do than the norm temporarily set. He should then be told to repractice those items that were incorrect. In this way the child determines whether he can work as rapidly as the standard indicated for his grade. He is stimulated to do better, if he possibly can, and is given opportunity to strengthen his specific weaknesses.

Much has been said in recent psychological discussions about overlearning. It is strongly favored by many because it gives the child a margin for forgetting. Overlearning on the whole seems to be a psychological invention of exaggerated significance. If the experience subjected to drill is really socially significant, the demands of daily life will insure further repetitions—normal and natural repetition. The school experience that must be overlearned probably occurs infrequently. Some of it, in fact, occurs so infrequently that we are hard pressed trying to justify the time and thought given to it in class. Thorndike believes in overlearning for he tells us:[1]

> Practice that is actually in excess is not a very grave defect if it is enjoyed and improves the pupil's attitude towards arithmetic. Not much time is lost; a hundred practices for each of a thousand bonds after mastery to 199 in 200 in 2 seconds will use up less than 60 hours, or 15 hours per year in Grades 3–6.

Further analysis of drill. In the discussion of standardized tests and in the teaching of each of the fundamental processes of arithmetic the reader will find additional material on various aspects of the drill lesson.

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[1] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), p. 137.

PROBLEMS FOR STUDY AND DISCUSSION

1. "Because there is considerable loss in carrying over facts learned in isolation, to real situations, formal drill may be eliminated and functional drill only should be used." Discuss the issues involved in this statement, define the necessary terms, and formulate your own position.

2. For each of the following arithmetical processes suggest an appropriate functional drill: (*a*) converting common fractions to decimal fractions; (*b*) reducing common fractions with dissimilar denominators to common denominators; (*c*) practice in column addition; (*d*) reducing denominate numbers of higher denomination to smaller denominations.

3. Give reasons and illustrations to substantiate the following:
   (*a*) Formal drills and isolated drills are not necessarily identical.
   (*b*) The drill designed to maintain a habit is different from the drill designed to initiate a habit.
   (*c*) Distributed learning insures the retention of newly acquired skills.
   (*d*) Motivation makes drill less arbitrary, but not necessarily pleasant.
   (*e*) "A skill that is practical in its natural setting will need less repetition."
   (*f*) In indiscriminate drill, every child wastes much time and energy.

4. Too frequently, drill degenerates into a form of busy work. Illustrate this statement. What steps would you take to provide helpful drill material? Think of (*a*) the specific skill to be acquired; (*b*) motivation and comprehension; (*c*) the functionalization of the practice. What other considerations will guide you in devising suitable practice material?

5. Prepare a bibliography of six items on the subject of this chapter. Consult the bibliography at the end of this book for guidance.

6. Examine critically a textbook in arithmetic to ascertain the provisions for drill. What is your estimate of the drill material provided?

7. Plan a new-type test on the subject, "Habituation in Arithmetic." Be sure to include true-false statements, completion and multiple-choice questions and matchings. Has your test covered the field? Does it call for memory responses only?

CHAPTER IX

DEVELOPING ACCURACY AND SPEED

Relation between accuracy and speed. Because of the nature of mathematics the desired standard of accuracy in arithmetical operations must be perfection. But rate of performance must also be considered because, all other conditions being equal, in any practical situation the more rapid worker has the advantage over the slower. It is desirable, therefore, to develop as high a rate of speed as is consistent with accuracy.

Investigations have centered around the relationship between speed and accuracy. Is the accurate worker accurate because he is slow or fast? Is the rapid worker less accurate or more accurate because he is rapid? In general we have enough objective data to establish the conclusion that there is a high correlation between speed and accuracy. To be sure, not all studies lead to the same conclusion. Courtis concludes, "Ignorance is shown, not by incorrect answers, but by reduced speed."[1] Phelps found that his slow group was most accurate and that the rapid group showed the greatest number of errors.[2] Monroe's studies warrant his concluding that the most rapid workers have the highest scores but in every group there are individuals whose speed is attained at the cost of accuracy.[3] Thorndike's statement is most positive: "The individuals who do most work in ten minutes will be above the average in a test in accuracy."[4] Monroe and Thorndike will therefore agree that any person who works at a rate distinctly above that to which he is accustomed will make more errors than he does working at his normal rate, but the *type of person* who tends to be more rapid is also more accurate.[5]

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[1] Quoted from Buswell and Judd, "Summary of Educational Investigations Relating to Arithmetic," University of Chicago, *Supplementary Educational Monographs*, No. 27, June, 1925, p. 53.

[2] C. L. Phelps, "A Study of Errors in Tests of Adding Ability," *Elementary School Teacher*, Vol. 14, September, 1913, pp. 29–39.

[3] W. S. Monroe, *A Report of the Use of the Courtis Standard Research Tests in Arithmetic in 24 Cities*, Bureau of Educational Measurements and Standards, No. 4 (Kansas State Normal School, 1915), p. 94.

From what has just been said it would not be fair to infer that stress should be placed on speed rather than on accuracy. Accuracy is certainly the more important factor in efficient work in arithmetic. Messick[6] gave two groups of children 4 minutes of drill daily for 20 consecutive days. Group I, the speed group, was told, "Add rapidly and think carefully as you work. We have found that as a rule the best adders are the ones who work most rapidly." Group II was told merely, "Add slowly and think carefully as you work. Be certain that each problem that you attempt is added correctly." Both groups gained, as we should expect, in both accuracy and speed. In Grade 4 the speed group improved 9 per cent more in speed but 44 per cent less in accuracy than the second group. The author's conclusions are worthy of note: "From the point of view of speed it makes little difference which is emphasized, speed or accuracy. From the point of view of accuracy, it is much better to emphasize accuracy." In addition, as in all basic processes in arithmetic, attention should be focused on accuracy rather than on speed.

The last word is yet to be said about the relations between speed and accuracy. We can safely summarize the known facts as follows: accuracy and speed are highly correlated, but making a given child work more rapidly will not necessarily increase his accuracy; while most people do not work at the highest rate, each person probably has an optimum rate which insures maximum accuracy for him.

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[4] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), p. 300.

[5] E. L. Thorndike, "On the Relation between Speed and Accuracy in Addition," *Journal of Educational Psychology*, Vol. 5, November, 1914, p. 537.

[6] A. I. Messick, "Effects of Certain Types of Speed Drills in Arithmetic," *Mathematics Teacher*, Vol. 19, February, 1926, pp. 104-109.

How can accuracy be increased? Accuracy can be improved by persistent emphasis on a few simple practices throughout the teaching of arithmetic. Irregular attention to these devices is productive of no results; regular insistence on these will gradually bear fruit.

1. *Develop habit in fundamental operations and combinations.* Our analysis of habit showed that habituated reactions are more accurate than those that result from deliberation. In all drills on basic operations and combinations in arithmetic the ever-guiding aim must be the development of a set of habits that will insure absolute accuracy.

2. *Legibility and placing of figures.* Much inaccuracy arises from the fact that children's figures are poorly formed or so carelessly or irregularly placed that correctness of final results is impossible. Civil-service boards throughout the country are now testing for this quality of penmanship in candidates for clerical positions. Legibility, when applied to figures or to penmanship, sums up a complex of qualities. It refers to "readability" under every variety of condition.

In multiplying by 204 we often find children extremely careless in the placing of the various figures of partial products.

$$\begin{array}{r} 736 \\ 204 \\ \hline 2044 \\ 147\ 2 \\ \hline 149964 \end{array}$$

In the accompanying illustration the 2 of the second partial product is erroneously added to the 4 of the first partial product; 9 is brought down and 2 and 7 give the second 9; the 14 completes the incorrect answer. Errors in every variety of number manipulation, especially in division, can be traced to similar carelessness of arrangement as well as to malformation of figures.

The conclusion for the teacher in all these observations is simple. Penmanship drills must include the writing of figures as well as of letters. But just as penmanship exercises give practice in writing words as well as letters, so, too, figures must be written in a variety of combinations and groupings so that the child develops ability to write figures legibly and to place them accurately. Scales for measuring penmanship should include specimens of figures in a progressive order of merit.

3. *Ability to copy figures.* Closely related to legibility of figures is the ability to copy figures accurately. Many teachers regard errors in copying figures as one of the chief sources of inaccuracy in arithmetic. These errors are usually one of three forms:

a. introducing new figures, that is, copying 2476 for 246
b. interchanging figures, that is, copying 37 for 73
c. omitting figures, that is, copying 37 for 307

Brown and Coffman[7] tell us that experimentally they have ascertained that the first type of error is more common than the second in the ratio of 28 to 11; that figures are omitted one-third as frequently as they are miscopied; that pupils below the average in speed miscopy more frequently than those above the average; that the lower the grade, the greater the number of errors due to miscopying. In addition to practice in the writing of figures regular exercises should be given in copying figures in a variety of groupings. There is little that the teacher can do to reduce errors in copying, aside from dignifying these exercises by showing their practical importance, giving them frequently, and ascertaining whether pupils who do not improve are suffering from some form of eye ailment. Well motivated, regular, frequent, and short practice lessons in copying figures correctly will help children to develop a desired degree of ability.

4. *Critical examination of all results.* The habit of testing

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[7] Brown and Coffman, "How to Teach Arithmetic" (Rowe, Peterson, & Co., 1925), pp. 54–55.

the result of a computation, so thoroughly ingrained in all accountants, must be developed in children. They must be taught how to approximate results and how to apply various checks. The important modes of testing answers and the system of checks applicable to each of the fundamental operations will be given in detail in a later chapter.

It is not enough to check the final answer. Children should get into the habit of checking each successive operation before beginning the next. In multiplying by 87 a partial product is obtained by multiplying by 7. Before beginning to multiply by 8 the first partial product should be checked by remultiplying. Similarly, before adding the two partial products, the multiplication by 8 should be checked. The earlier an error is discovered the less trouble it causes.

5. *Habit of planning work and seeking correct interpretation of problems.* An untrained mind usually attacks a problem in feverish haste without attempting to familiarize itself with the facts and conditions of the situation or to understand clearly what is demanded. The general appearance of the problem may recall a seemingly similar type, and the solution is often half completed before the child realizes that new conditions have been introduced. Frequently the confusion that results is too great to permit a clear rethinking and a successful second attempt. Children must be taught how to analyze problems and plan for their solution. The specific method and devices for teaching children modes of attacking problems must be postponed to Chapter XXXII, "Problem-Solving."

6. *Accuracy of statement.* We must reëmphasize the fact that accuracy of statement makes for accuracy of thought and that looseness of statement encourages looseness of thought. The suggestions embodied in the discussion of accuracy of algorism and precision in the use of language in arithmetic may be applied with equal force at this point. (See page 79.)

7. *Practice to be based on diagnosis of individual needs.* As we have seen, the old adage "Practice makes perfect" is not without its serious limitations. Not all practice makes

perfect. Unless teachers use diagnostic tests and ascertain individual needs too many pupils are practicing much that they know. Practice with differentiated material based on individual needs motivates itself to the child and arouses in him reactions that make for intelligent and accurate responses. Diagnostic testing and remedial teaching are discussed at greater length in later chapters.

How to increase speed? Speed is influenced by many factors, of which native ability is perhaps the most important. The more intelligent pupils are usually the more rapid workers. The attitude of the individual pupil is almost as important: indifference or a will to do determines both the rate and the quality of the reactions. Conditions imposed by unwise classroom management may reduce rate. For instance, a fixed arrangement of work may be required, answers not placed in a designated part of the paper may not be acceptable, too many figures may require labeling, too many numbers may have to be copied, or an unusually high quality of penmanship may be maintained. These conditions worry children and therefore obtrude themselves too prominently into arithmetic thinking. Where artificial conditions do not interfere with children's natural reactions much can be done to quicken responses.

1. *Develop accuracy.* Although it is true that the accurate worker in arithmetic is not always rapid it is axiomatic that inaccuracy is the greatest obstacle to speed. The plea made for one of our school systems—that even though it ranked below average in accuracy it nevertheless stood much above average in speed—did not impress one of the school superintendents who saw in the explanation only that it "takes us less time to get a thing wrong than it does in the average school system." The habit of accuracy is perhaps the greatest single factor that makes for speed.

2. *Acquisition of necessary habits in arithmetic.* Reactions that are automatic are always more rapid than those that occur through direction by higher centers. The more effort intelligently spent on inculcating in children the com-

plete round of necessary subject habits and operation habits, the more is the teacher contributing toward increased speed of the pupils.

3. *Limit time for assigned work.* Experience shows that the most rapid workers in arithmetic have developed speed as need for it became apparent. The inexperienced bill clerk or bookkeeper finds that the conditions of business demand more rapid computation. He puts an extra strain on himself, gives himself more completely to the task at hand, and learns that he can increase his rate of work. It is wise, therefore, to limit the time for all assignments in arithmetic. After the task is set the children should be told that they will be held responsible for its completion at the end of a given period. The class applies itself with greater concentration. To allow children to take the time that they deem necessary may discourage concentration and dissipate effort. In all tests the teacher's records of former pupils' performances and not the class, should determine the amount of time required for their completion.

4. *The use of short cuts.* Much time can be saved by teaching children to use short cuts when longer processes are understood. When short cuts are used—not at the teacher's direction, but from force of habit—economy of time and effort is achieved.

5. *Reduce formal analysis.* A cause of much of the slow, cumbersome solution of problems can be traced to excessive analyses and algorisms with their many labels and statements. We must look upon formal analysis as an aid to systematic thinking when a type of problem is new. But just as soon as children give evidence of a thorough comprehension of the thought involved, formal analysis must be abandoned. It must be our constant aim to reduce to an absolute minimum the language, oral and written, accompanying the solution of a problem.

6. *Use a minimum of figures.* Teachers must also strive to have their pupils use as few figures as possible in the solution of a problem. The problem given to a sixth year class

was: "The census shows that 55 per cent of the population of a city of 145 000 inhabitants is colored; 20 per cent of the Negro population is in skilled trades; how many Negroes have skilled trades?" The model solution that the teacher required was as follows:

$$\begin{array}{r} 145\,000 \text{ inhabitants} \\ \underline{.55\phantom{\text{ inhabitants}}} \\ 725\phantom{00\text{ inhabitants}} \\ \underline{725\phantom{000\text{ inhabitants}}} \\ 79\,750.00 \text{ Negroes} \end{array}$$

$1/\cancel{5}$ of $\overset{15\,950}{\cancel{79}\,\cancel{750}}$ people = 15 950, Negroes in trades. *Answer.*

Judging by the number of children who obtained the correct answer there was no need of insisting on so many labeled numbers or so careful a statement of each operation. The following is a more economical solution:

$$\frac{1}{\cancel{5}} \text{ of } \frac{\overset{11}{\cancel{55}}}{1\cancel{00}} \text{ of } 145\,0\cancel{00} = 15\,950, \text{ Negroes in trades.}$$

The time saved in solving this one example by the shorter method is significant; but guarding the child from developing habits of work that require needless penmanship and useless repetition of numbers is even more important.

7. *Assign work to children according to their ability.* The practice of assigning a fixed number of examples to all children in a class is responsible for the development of habits of slow work among many able pupils. When uniform assignments are made without regard to pupils' ability, the more capable children complete their work in a relatively short time. Some of these pupils work rapidly for the satisfaction they feel in finishing first, while others soon learn that even though they work slowly they nevertheless complete the given task in good time. Gradually they acquire habits of divided attention, which cling with a persistence that may render futile their later endeavors at concentration. The abler children should have more to do in a given time than their less capable classmates—more quantitatively and, surely, more exacting intellectually.

8. *Appeal to pupils.* The most direct means for increasing speed in arithmetic is probably the most effective. Children should be made to understand the importance of learning to work at one's maximum rate. Individual records of rate, kept by each pupil, help to stimulate children and to interest them in satisfactorily completing a task in a minimum of time.

Problems for Study and Discussion

1. Two trained investigators conduct similar experimental studies in the relation between speed and accuracy, and come to conclusions that are decidedly different. Account for the difference in result.

2. A teacher of a homogeneously grouped class is eager to ascertain the degree of accuracy and speed which her class should attain in arithmetic. Where should this teacher look for the standards or norms she may reasonably expect her pupils to attain?

3. A father finds that his child does not know answers to 5 + 7 and 9 + 8. He helps the child as follows: "5 + 5 = ? 7 is 2 more than 5; therefore 5 + 7 = ?" Says the child, "5 + 5 = 10; 10 + 2 = 12; 5 + 7 = 12." The father further explains: "You don't know 9 + 8. Do you know 10 + 8?" The child says "18." "But," continues the father, "10 is how many more than 9? Therefore 9 + 8 is not 18 but —?" The child says "17." If the teacher becomes aware of this incident and has an opportunity to talk matters over with the parent, what should she say? Should she praise or condemn the father's method? Why?

4. Compare the two sets of suggestions for increasing speed and accuracy. Point out the duplications and account for them.

5. Can you add to the suggestions listed here for increasing speed and accuracy? Do you take issue with any of the suggestions? Why?

CHAPTER X

RATIONALIZATION AND OBJECTIVE AIDS

Importance of the problem. An ever-present problem in the teaching of arithmetic is the determination of the extent to which number facts and processes should be based on reason. Much time is often spent explaining to children what, in their immaturity, they cannot understand. Rationalization of arithmetical processes may be an expression of an adult interest in the subject rather than an answer to the child's curiosity. On the other hand, to eliminate all rationalization produces thorough mechanization. The resulting instruction is deadening and robs the subject of every vestige of disciplinary worth. A plan of instruction in arithmetic that sets intelligent limits to the process of rationalization not only achieves economy of time and energy, but also retains the mental vigor which should accompany the study of elementary mathematics.

Older modes of instruction in arithmetic were characterized by severe memory drills and constant imitation of model solutions. Facts were accepted on the teacher's authority and were at once applied by rule to a variety of situations. Thorough habituation was the aim; blind rote work seemed to be the only means of attaining it. The reader may have a vivid recollection of memorizing the law, "To divide a fraction by a fraction, invert the divisor and proceed as in multiplication." After this stereotyped sentence was learned it was explained and applied to a few examples by the teacher. The class then followed the mode of solution by finding answers to other examples.

The reaction came with the popularization of methodology. Deductive teaching was condemned; inductive teaching, the development method, and objective demonstrations were extolled. Every step was explained, every process rationalized, every relationship discovered anew by the children, and every number properly labeled. Children were so occupied with the *why* that they often lost the *how* and the *what*. The elaborate devices of bundling sticks into tens and hundreds and of packing beads into bags of tens, hundreds, or thousands in preparation for the teaching of notation to seven-year-old children are illustrations of the type of method prompted by the fetish of rationalization.

To-day there is a clear realization that there are limits to both rationalized and mechanized teaching not to be transcended; that there is need of differentiating mechanical from rational processes in arithmetic; and that the former must be reduced without rationalization to habit, while the latter may be accorded a reasonable measure of rationalization.

Those who set themselves against all rationalizations, except in the solution of problems, seek to emphasize arithmetic as an art rather than as a science. Their position appeals to many teachers and must therefore be analyzed and evaluated critically. We present here no one person's position but rather opinions held by a group of earnest teachers.

Arguments of Opponents of Rationalized Arithmetic	*The Criticism*
1. What is to be used mechanically in everyday life must be taught mechanically in school. All the processes of arithmetic are applied in post-school days from force of habit and should therefore be taught by a series of repetitions without rationalized explanations.	1. The argument is a *non sequitur*. The aim in all teaching is to mechanize all necessary knowledge so that it is reduced to habit and frees the mind for new adjustments. The products of the United States or the facts of the most important tables of weights and measures should be so thoroughly mastered that they are recalled when necessary without appreciable conscious effort. But we do not counsel teaching a list of products without leading children to understand the physical forces which bring them

Arguments of Opponents of Rationalized Arithmetic	*The Criticism*
	about. Nor do we tolerate methods of teaching which impose a table without having children first see the units and discover the various relationships. The present tendency in teaching is to proceed from rationalization to mechanization.
2. Since habit is the final goal in all instruction the teacher must impose on her pupils a generous round of practice. To rationalize is to focus consciousness on a reaction that should become automatic.	2. True, habits make for mental economy, but unless habits are preceded by appeal to reason the individual is helpless when new conditions arise. To habituate without understanding reduces life to stupid repetitions utterly dependent on the avoidance of change.

The bond theory, frequently misapplied, leads many to assume that Thorndike would teach each number combination as a discrete bond without any justifying reason.[8] But Thorndike protests against any such interpretation, for he says, "I hasten to add that the psychologists of to-day do not wish to make the learning of arithmetic a mere matter of acquiring thousands of disconnected habits nor to decrease by one jot the pupils' genuine comprehension of its general truth." [9]

3. Efficiency is not impaired if children do not know the justification for a process they perform automatically; why, therefore, "stir up a hornet's nest by explaining borrowing in subtraction, carrying in addition, or the mode of building up tables of multiplication or of weights and measures?"	3. As long as there is no variation from the type, efficiency in a process may not be determined by comprehension of it. But new conditions arise constantly in arithmetic, even in the mechanical operations. A pupil who understands a mode of solution applies what he knows when the type form is changed. Rationalization gives an intelligent insight into a process and enables the pupil to meet new conditions.

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[8] William A. Brownell, "The Development of Children's Number Ideas in the Primary Grades," *Supplementary Educational Monographs*, University of Chicago, Bulletin 35, August, 1928, pp. 195–215.

[9] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), p. 73.

Arguments of Opponents of Rationalized Arithmetic	*The Criticism*
4. Economy of time demands an elimination of all attempts at rationalization in every operation in arithmetic.	4. Immediate economy of time results from the elimination of all explanations and from the mechanical imposition of knowledge. But in the long run this economy is dissipated by the helplessness which such methods develop. If all geography and history were taught through repetition without explanation, in order to establish fixed associations, even more time would be saved, but in the long run the loss of power would far outweigh the gain in time.

Principles that govern rationalization in arithmetic. 1. When an arithmetical process occurs in variable form it is necessary to rationalize it. An analysis of the suggested gradations found in the treatment of the various processes will give the reader numerous examples.

2. When there is doubt about the children's ability to comprehend underlying reasons it is wise to omit all rationalization. Any attempt to explain to slow fifth-grade children why increasing the denominator decreases the value of a fraction, or why decreasing the denominator increases the value is usually doomed to failure.

3. When repeated use and increasing maturity will reveal the reason for a process that cannot be explained early in the school course without the expenditure of much time and effort it is advisable to postpone the rationalization. Division of fractions, multiplying a quantity by a fraction whose value is less than one, division by numbers as large as 24—these are illustrations of arithmetical operations that may be presented without rationalization, for they acquire meaning through extended application and the more mature reason of later years.

4. The same standard of rationalization must not be applied to all children. Those who give evidence of penetration in

quantitative relations may be led to see reasons in processes that must be taught without justification to others less gifted. To some children rationalization is a source of confusion; to others it brings interest and gives a deeper grasp of the essential factors.

No hard-and-fast rules can be laid down to govern all classes, all children, or all topics. The teacher must make the adjustment that seems most appropriate in a given situation.

Transition from rationalization to habituation. In concluding the discussion of rationalization in arithmetic we must note the stages by which a process is reduced to habit. These may be summed up under four steps:

1. The teacher demonstrates the process at the blackboard. Questions are asked and the children's coöperation is enlisted as the explanation is evolved. Throughout this rationalization the teacher may solve the illustrative problems alone. The object of this initial step is to *insure comprehension* by the children.

2. The pupils are now required to apply the new law of arithmetic to problems, as many as possible working on the blackboard. The teacher scrutinizes the work, points out individual and general errors, and encourages the hesitant pupil. The aim is now to *clarify the children's ideas.*

3. As the progress of the pupils warrants, seat work is introduced. Examples are assigned for solution by the pupils at their desks. The teacher maintains supervision which is rather indirect and concerned primarily with the weaker pupils. The purpose now is *to afford drill through variety of application* so that the process of habituation is set in active motion.

4. The final stage is characterized by assignments for home work or study periods. The object is now to secure *independent application.*

These four steps in the transition from rationalization to habituation may be modified, enriched, or contracted as changing conditions suggest. Only when they are used with this freedom do they serve classroom purpose well.

Thorndike believes in giving explanations and reasons of arithmetical processes, yet is inclined to accept the position that to teach a reason that is bound to drop out of sight and mind entails serious waste. He argues that the reason be kept alive through use in checking or proving one's work and through use in explaining other procedures. In teaching division by a decimal he insists on the explanation—multiply both dividend and divisor by a number which makes the divisor a whole number. This the child keeps applying as the examples change; 24.32÷.004, 24.4÷.04; 2432÷.04. In gradation of subject-matter—say in subtraction of fractions—the topic grows, bit by bit, habit upon habit, until a whole series of progressive habits is acquired. But there is also growth in understanding of arithmetic because each habit is based on an intelligent insight into number.

> Almost nothing in arithmetic need be taught as a matter of mere unreasoning habit or memory, or need anything, first taught as a principle, ever become a mere matter of habit or memory. . . . Almost everything in arithmetic should be taught as a habit that has connections with habits already acquired and will work in an organization with other habits to come. The use of this hierarchy of habits to solve novel problems is reasoning.[10]

Objective aids. *When to be objective.* The observer of classroom instruction in arithmetic is often forcibly impressed with the unorganized and ill-planned objective teaching in the early grades, and with its thorough elimination in the upper grades. This condition is due to the naïve assumption that the young child is in a "sense-perception stage" while the older pupils are in the "conceptualization stage." For this reason there is a superfluity of games, objects, pictures, and diagrams at every step in the work of the introductory stages, and an insistence on the memorization and mouthing of imperfectly understood generalizations in later years. We must realize that all primary ideas, all new concepts, must be built up through concrete and motor experiences irrespective of the age of the student. The pupil, who in the last two years of his school course is studying the metric system, must be taught these new units and their relations in as objective a manner as the table of linear measure was taught him in a very early grade. Not the age of the students but the newness of the experience determines the degree of objectiveness in teaching.

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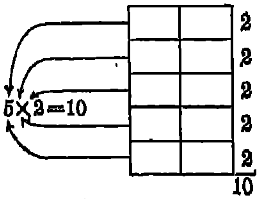
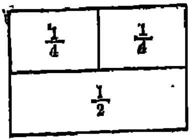
[10] E. L. Thorndike, *Psychology of Arithmetic* (Macmillan Co., 1922), p. 194.

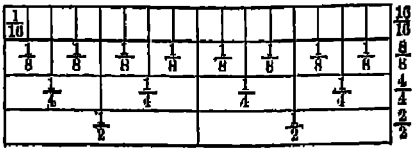
Ineffective objective material. Diagrams like the following abound in textbooks and fill the blackboards during recitations in arithmetic. Let us examine them critically in order to ascertain whether all that appeals to the senses is really an objective aid to teaching. The student of teaching methods will undoubtedly grant that these diagrams have little value in teaching. They are static and convey little or nothing to the immature child. They are symbolic representations, not concrete presentations of experience. To us who are familiar with the primary facts of number, these diagrams summarize graphically the relations among halves, quarters, and eighths. To children still groping in strange territory these diagrams with their divisions and subdivisions present additional symbols to be learned.



To demonstrate 5×2 is the same as 5 two's



To demonstrate that $\frac{2}{4}=\frac{1}{2}$

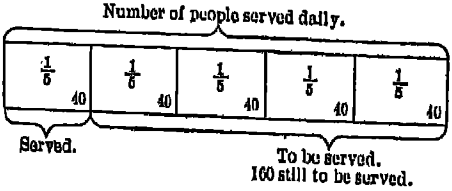


To demonstrate that $\frac{2}{2}=\frac{4}{4}=\frac{8}{8}=\frac{16}{16}=1$

Let us examine another illustration. In schools that teach the indirect cases in fractions or percentage the following illustration is still regarded as helpful to explain the following problem:

> A milkman who just delivered milk to ⅕ of his customers, still has 160 people to serve; how many people does he serve daily?

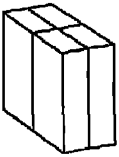
What are the children taught to do? To draw a large rectangle

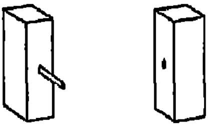


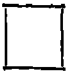
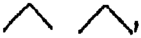
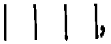
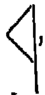
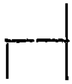
to represent all the people served daily; then ⅕ of the rectangle represents the number already served on this particular day. Then it follows that ⅘ or 4 sections represent the number still to be served. But we know that 160 are still to be served. Hence ⅘ of those daily served are 160 people and ⅕ of them must be 40. The five fifths are therefore 5 times 40 or 200. But in the final analysis this is not a concrete presentation but a *symbolic representation,* for the long rectangle *represents* one quantity, each part *represents* another and the grouping into four-fifths and one-fifth *represents* a quantitative relationship. The diagram above does not make the problem any more concrete than does the diagrammatic representation of the grammatical parts of a sentence. The term *concrete* as used in teaching is almost technical. It means more than the dictionary definition, "Anything that can be sensed." It refers to something that is familiar, rather than something that can be sensed.

Games have been given a prominent place in the teaching of arithmetic. The purpose is clear and laudatory but it is defeated by the uncritical use of games. We must be sure that the game is a true game, not a game in name only; that the game is so simple that attention is not diverted from number relations to rules of playing; that every step in the game involves number facts which the child is learning, incidentally, to be sure, but learning none the less; that the game does not take an inordinate amount of time. In a later chapter we shall discuss the subject of games in greater detail.

Effective objective material. The most helpful illustrative material is the kind that may be used directly and that lends itself to very ready manipulation. It can be changed and rearranged with ease. Hence not elaborate diagrams, but simple materials and activities like the following should be favored: blocks, coins, bell rings, foot rules, grouping children, toothpicks, marbles, panels in the blackboard, dominos, window-panes, lights in a cluster, units of measure like pound, yard, cent, or dime, arrangement of seats and desks, and paper-folding. A set of building forms in which the uniform unit can be separated into fractional parts is infinitely superior to cutting the traditional pie or the apple to teach $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$. Figure A represents the whole unit; figure B, its parts.


A


B

The child who actually takes these apart and puts them together again learns relationships among the parts of a unit with an effectiveness that no static diagram can insure. What in the concept *fraction* cannot be learned by paper folding? With the aid of four toothpicks the child learns all the ratios of four. He is asked to arrange these four sticks to make a box , two Indian tents,  four Indians, , a weather flag, , and a chair, . This variety of arrangement teaches the child that 4×1, 2×2, 1+1+1+1, 3+1, and 1+3 all equal 4.

What is very well known is as objective, at least, as diagrammatic representations like dots, rectangles, and circles. For illustrative material we must draw heavily on familiar occupations, games, measurement of materials in manual work, and on the record-keeping incidental in classroom, domestic, and recreational experiences. We must ever be mindful of the fact that static and uncommon objective material is as unobjective as mere verbal presentation.

Can objective teaching be overstressed? We have already emphasized the importance of teaching a new concept objectively. To continue to objectify concepts that are no longer new weakens thinking by denying it the opportunity to deal with the abstract. A growing mind exhibits increased power to transcend the confining bonds of concrete experiences. To continue objective teaching beyond initial stages in new topics is to rob arithmetic of its disciplinary value. Objective teaching usually runs to extremes—it is either neglected or made a fetish.

PROBLEMS FOR STUDY AND DISCUSSION

1. Devise twelve true-false questions, six completion questions, and six multiple-choice questions designed to test your classmates' grasp of the subject of rationalization in arithmetic. Compare the scope of your twenty-four questions with the scope of the discussion of rationalization in this chapter.

2. What common ideas do you find in this discussion of rationalization and in the earlier discussion of habituation through drill?

3. List five arithmetical processes or skills that you would teach without rationalization. Give your justification in each case.

4. List three arithmetical processes or skills that you would have children habituate first and rationalize later. Give your justification in each case.

5. Indicate which are true and which are false. Give reasons for your decisions.

(*a*) Immature pupils may be permitted to habituate a necessary process before it is rationalized.
(*b*) What is to be applied habitually later, should be learned without rationalization.
(*c*) Functional drills do not develop habits as readily as formal drills.

(*d*) A process may be rationalized for some pupils, but not for others in the same class.

(*e*) No arithmetic in the first four years should be rationalized.

(*f*) Mature people of superior intelligence find difficulty in rationalizing an arithmetical process which they perform with absolute accuracy.

6. "Realities must be associated with every symbol in the mind of the child; if continued too long, the concrete becomes burdensome; if not used wisely when necessary, number becomes meaningless manipulation." (Quoted in *Contemporary Guide in the Teaching of Arithmetic,* p. 1.)

Identify all the facts offered in this chapter in support of each of the three clauses in this quotation.

PART IV

IMPROVEMENT IN THE TEACHING OF ARITHMETIC THROUGH CONSTRUCTIVE SUPERVISION

CHAPTER XI

SUPERVISION OF THE TEACHING OF ARITHMETIC

The effectiveness of teachers' work is conditioned, in no small degree, by the quality of supervision. No teacher, however extensive her experience, can serve her pupils to her fullest ability without supervision; at best she is but a single agent in a coöperative group—the entire school staff. Objectives must be set up, the aims of the different subjects must be harmonized, course-of-study requirements must be interpreted, results must be evaluated, and methodology scrutinized in the light of our philosophy of education and our knowledge of psychology. The influence of the supervision, indirect though it may be, is potent and may be exercised in a variety of ways. We must now consider some of the important supervisory aids.

Supervision a coöperative enterprise. The prevalent conception of school supervision is too often borrowed from the shop and the counting house. In business enterprises responsibility is highly centralized in a single person or in a governing board. Authority is then delegated to subordinates who in turn delegate part of their authority to their subordinates. The result is a system of supervision in which each person is definitely responsible to a particular superior, who in turn is responsible to his superior, until the apex of pyramided responsibility is reached. Direction in business is from above. Subordinates try to please their superiors, who expect loyalty and obedience from them at all times.

This is not the place to challenge the wisdom of the prevailing system of business supervision. But we do protest against its introduction into an educational enterprise. In a school supervision must be a coöperative activity, a coöperative search for the most effective means of helping immature minds attain their full realization. No school supervisor, however able, can safely rely solely on direction which he can exert from above. His chief duty is to help his teachers recognize the unscientific character of much current educational procedure and to make them professionally minded. Then will they become critical of their own teaching practices and seek more effective teaching techniques.

Supervision through conferences. The supervisor is charged, in most school systems, with holding stated conferences with his teachers. Too often these meetings degenerate into dictation of specific administrative regulations and of teaching practices. Once more the system of supervision in business is imitated.

A teachers' conference must reflect the spirit of coöperative supervision to which we have referred. The topic of the conference should be closely related to school needs, and if possible should be selected from among the many problems suggested by a committee of teachers charged with responsibility for the conference.

The call for the conference should precede the conference day by at least a week, should state very clearly the conference subject, and should suggest preparation for it that is not too time-consuming. Teachers are in closest relation to the learner and should be able, therefore, to make helpful contributions to the conference. Frequently teachers attend a conference without any preparation—without even knowing the subject to be discussed. Small wonder that under such circumstances the end of the day finds teachers too tired to enter vigorously into the conference. They sit back complacently and listen to their supervisor. As an example of a helpful conference call we may suggest the following:

February 1.

The next conference is scheduled for February 15, at 3:10, in Room 430. We shall continue the subject of individualization of instruction. The specific topic for the day will be "Differentiated Drills in Arithmetic." Teachers are urged to reread carefully the sections of the course of study related to the topic and to present to their grade chairmen specimen drills they have used to meet individual pupil needs. Chairmen will be prepared to report helpful devices to the conferences. Your Committee on Conferences suggests that the discussion be limited to the following questions:

1. To what extent can drills be differentiated? For every pupil or for a limited number of groups?

2. On what bases should children be grouped in the several grades for differentiated drills?

3. How should the differentiated practice material be presented?

4. How much time shall be devoted daily to differentiated drill? In class? At home?

5. How shall the children check their progress?

6. How shall teachers supervise this type of class work? Helpful suggestions will be found in the following references which are available in the school library: (1) ____________; (2) ____________; (3) ____________.

(Signed) ____________________ Principal

The supervisor may preside at the meeting, or not, as he sees fit. He may be an auditor, a speaker, a leader in the discussion, or the final speaker who organizes, summarizes, and evaluates the data presented. The principal should not always be the chairman nor ever be the sole and final arbiter of moot questions. Much should be referred to standing committees for special report.

A conference discussion in which many teachers are active participants should have a permanent secretary who makes record of suggestions, of agreements reached in actual vote, of questions referred to committees, of committee reports, and the like. Mimeographed summaries of these conference meetings serve as helpful records for the staff.

Unless there is systematic follow-up, the conclusions and agreements of such meetings may not be translated into class practice. The supervisor whose conferences are conducted along these lines can make demands on his staff with good grace, for he is asking adherence to the decisions of the staff rather than faithful obedience to a program which he has arbitrarily laid down.

Visiting other teachers. As a rule teachers know only their own teaching techniques because they do not see their colleagues at work. It is important that interclass visits be arranged. But these visits must not be made at random. Supervisors must decide what teachers should be visited and what teachers need the stimulation and the guidance which come from observing a fellow-worker. The teacher who is asked to visit her colleague should either be told in advance what to look for or be guided, after her visit, in evaluating what she observed. It is not uncommon to find that each of four teachers of a given grade, or of closely related grades or subjects, has something distinctive to offer her colleagues. Under such circumstances interclass visits are less likely to arouse resentment and breed misunderstanding than when only one or two teachers are pointed out as possessing procedures worthy of being adopted. We are assuming, of course, a tactful and discriminating supervisor.

Model lessons. An occasional lesson taught by a supervisor, on the impulse of the moment, is more helpful to the supervisor than to the teacher. A model lesson presupposes frequent preliminary visits in which the supervisor becomes familiar with a teacher's professional shortcomings. The teacher must know the definite reason which leads the supervisor to demonstrate a type of teaching. The lesson must also have a specific aim for the pupils and must be given within the time limits set for the teacher. After the model lesson time must be found for a conference with the teacher. Unless she is convinced of the superiority of the model lesson over her own teaching, little good is accomplished and much hostility is aroused. Supervisors must be reluctant to meet every request of their teachers for a model lesson. As a rule, a selected teacher should give the lesson and in the discussion that follows, the superiority of the demonstrated teaching can be effectively brought out.

The use of inventory and diagnostic tests. In a later chapter we shall discuss in full the values of standardized tests for constructive supervision. It will suffice, at this point, to indicate that a test which is truly diagnostic gives the teacher an accurate basis for differentiated drill and the pupil an adequate motive for continued practice. The inventory tests that carry out the promise in their name have their obvious values for improvement of instruction. Unless they have a clear knowledge of what children know and need to know, supervisors and teachers are badly handicapped in formulating the materials and methods of instruction.

PROBLEMS FOR STUDY AND DISCUSSION

1. List the specific aids to instruction that may be derived from constructive and coöperative supervision. Have you ever received any of these aids from supervisors whom you know? What reasons account for the gap between what is possible and what is actually achieved?
2. This chapter discusses the means by which a supervisor may guide teachers to improve their teaching techniques. Arrange these means in order of effectiveness. On what principle or principles are your decisions based?
3. If you have never attended a teachers' conference, try to secure an invitation to one. Observe critically all that takes place. What part did the teachers have in deciding on the subject of the conference? When were they apprised of what would be discussed? What part did the supervisor take in the conference? In what way did the teachers participate? Whose decisions prevailed? What was the attitude of the teachers toward the conference? Indicate, the changes, if any, that you would make in the conduct of a conference.
4. A lesson, taught by the supervisor, or by a selected teacher for the benefit of a single teacher or a group of teachers frequently arouses misunderstanding and resentment. Why? What cautions may be exercised to minimize or eliminate this hostile attitude?

CHAPTER XII

INFORMAL TESTS IN ARITHMETIC

We shall not discuss here those aspects of testing or examining which are usually treated in books on class management. Nor shall we develop at this time the subject of standardized tests because we shall devote a whole chapter to it later. We shall analyze diagnostic testing in connection with the teaching of each of the basic operations. We are concerned now with the informal tests made by teachers at the conclusion of an important topic. Where the practice is to follow, seriatim, a favorite textbook or drill book, teachers are relieved from the task of devising such tests because these books provide them. When these do not appear to be adequate measures of progress, teachers must supplement them by their own tests.

Objectives of examining in arithmetic. There are two sets of reasons that make stated examinations desirable as an integral part of instruction in arithmetic. These reasons are the chief concern of the pupil and of the teacher, and group themselves, therefore, under two heads.

Outcomes concerning the child. From the point of view of the pupil, the examination may be given (*a*) to lead the child to discover his weakness and thus to exert greater effort to improve in special fields; (*b*) to stir pride in the progress indicated by a high rating, and thus feel encouraged to continued endeavor; (*c*) to serve as a motive for reviewing the work of a part of the term; (*d*) to develop the habit of self-correction.

Outcomes concerning the teacher. To the teacher the occasional test serves very different but equally useful purposes.

Teachers must know the status of their pupils' knowledge before deciding whether to teach new facts and skills or provide continued practice on what was taught. Individual weaknesses must be ascertained periodically in order to give the teacher a basis for devising corrective measures and differentiated practice material.

During the correction of a carefully planned test a teacher has ample opportunity to gauge the merit of her instruction. She can tell what was adequately taught and what needs reteaching. Let us assume that she taught reduction of common fractions to lowest terms. In the test the children's work shows that they use any common divisor instead of trying to find a reasonably large one or the largest one, thus

$$\frac{12}{16} = \frac{6}{8} = \frac{3}{4} \qquad \frac{18}{24} = \frac{9}{12} = \frac{3}{4} \qquad \frac{12}{18} = \frac{6}{9} = \frac{2}{3}$$

That the children know how to apply the rule of dividing numerator and denominator by the same number is obvious. That the teacher did not stress the advisability of using the largest divisor, not the first that comes to mind, must be admitted. Such test findings justify the test.

Need of aims in testing as well as in teaching. There is as much need for a specific aim in a well planned test as in a well conceived lesson. A test must be designed to disclose individual weaknesses and the degree of mastery of *specific* techniques of arithmetic. Ten examples do not necessarily constitute a test. What if one is on column addition, one on finding an area, one on division, one on decimal notation, one in addition and subtraction of fractions, one in division by a fraction, one in reduction of denominate numbers, and three in problems which call for finding the whole when a fractional part is given? Such a composite test has little pedagogical significance for it conceals more than it reveals. Let us assume that a child scores 60 per cent in such a test or that a class averages 55 per cent. What inference can be drawn that will be useful in the continued teaching of that class? Such a composite examination paper may have its place at the end of a term when a rating or final grade is required, but as a test it is devoid of educational significance.

In organizing a test a definite aim must be formulated so that the questions bear directly on one objective. An examination may be given to determine ability in any one of the four fundamental operations, in multiplication and division of fractions, in decimals, in copying figures accurately, in the use of tables of weights and measures, in speed of manipulation, in problems that follow a type, or in solving problems that vary from that type. When the teacher is testing for accuracy, no credit is given for wrong answers obtained by model solutions; when the test is on reasoning ability, due credit is given for an intelligently planned solution, even though the result is incorrect; where the test is to ascertain the degree of speed, the time must be limited. Such specialized and differentiated tests reveal the weaknesses of each individual, indicate where further drill is needed, and show what changes must be made in the course of study. Illustrations of tests in arithmetic that lead to constructive conclusions and that are scientific in their attitude, if not in their construction, will be found in the chapter on standardized tests and in the various sections dealing with diagnostic testing for common errors and their corrective measure.

Types of questions. In addition to the usual questions found so frequently on teacher-made tests we may employ the new-type questions so common in the content subjects. A test made up in large part of the new-type questions covers a large field of knowledge and many skills in limited time, and gives the child practice in new modes of thinking about his arithmetical experiences. The various types of these questions, together with appropriate illustrations, are given below. A word of caution, however, is necessary. These illustrative questions of any one type are entirely too few to constitute a whole test.

True-False Questions

Topic: Reduction of Common Fractions to Lowest Terms

Before each statement is a blank line. In it write *True,* if the statement is correct, and *False,* if it is incorrect.

________1. The word, *reduce,* in the expression, reduce to lowest terms, means, make the value smaller.

________2. If a numerator of a fraction is divided by four, its denominator must also be divided by four.

________3. In reducing $\frac{8}{12}$ to lowest terms, the best number to use as a divisor of both numerator and denominator is 4.

________4. If both numerator and denominator of a fraction are even, they can *always* be divided by any even number.

________5. $\frac{3}{12}$ is a fraction in its lowest terms.

Multiple Choice

Topic: Decimal Notation

In each statement you will find four answers but *only one* is correct. *Underline the one that is correct.* Do not guess.

1. The nearest in actual money to .199 is (*a*) 19 cents; (*b*) 20 cents; (*c*) one dollar and 99 cents; (*d*) 199 cents.

2. .33$\frac{1}{3}$ is equivalent to (*a*) $\frac{1}{3}$; (*b*) thirty-three hundreds; (*c*) $\frac{3}{10}$; (*d*) $\frac{33\frac{1}{3}}{100}$.

3. .00$\frac{1}{2}$ is equivalent to (*a*) .05; (*b*) .050; (*c*) .005; (*d*) $\frac{5}{10}$.

4. 124.006 should be read as (*a*) one hundred twenty-four and six thousandths; (*b*) one hundred twenty-four and six thousands; (*c*) one hundred and twenty-four and six thousands; (*d*) one hundred and twenty-four and six thousandths.

5. .0$\frac{1}{8}$ may not be written as (*a*) .01$\frac{1}{4}$; (*b*) .012$\frac{1}{2}$; (*c*) .0125; (*d*) .12$\frac{1}{2}$

Completion Questions

Topic: The Function of Numerator and Denominator

Each statement needs one word to make it both complete and correct. Supply that word in each statement and write it on the line provided for that purpose.

1. The greater the numerator, the ____________ the value of the fraction.

2. The greater the denominator, the ____________ the value of the fraction.

3. Dividing the denominator ____________ the value of the fraction.

4. Multiplying the denominator ____________ the value of the fraction.

5. Multiplying both numerator and denominator by different numbers ____________ the value of the fraction.

Matching Forms and Values

Topic: Fundamental Operations in Common Fractions

In the first column you will find a set of examples. Somewhere in the second column you will find the answer. For example, if the answer to (*a*) $\frac{1}{3}+\frac{1}{9}$ is $\frac{1}{27}$ then write (*a*) before $\frac{1}{27}$ in the second column.

(*a*) $\frac{1}{3}+\frac{1}{9}$
(*b*) $\frac{1}{3}-\frac{1}{9}$
(*c*) $\frac{1}{3}\times\frac{1}{9}$
(*d*) $\frac{1}{3}\div\frac{1}{9}$
(*e*) $\frac{1}{9}\div\frac{1}{3}$

__________ 3
__________ $\frac{4}{9}$
__________ $\frac{1}{3}$
__________ $\frac{2}{9}$
__________ $\frac{1}{27}$

Labeling

Topic: Terminology in Division of Common Fractions

Consider the example $\frac{5}{8}\div\frac{1}{4}=\frac{5}{2}=2\frac{1}{2}$. Work it out in the margin and obtain the answer $2\frac{1}{2}$. Now write the correct word in each space of each statement.

1. 5 and 1 in the fractions $\frac{5}{8}$ and $\frac{1}{4}$ are called ____________.
2. The denominators in these fractions are ______ and ______.
3. The dividend is ________.
4. The divisor is ________.
5. The quotient is ________.
6. $\frac{5}{2}$ is called an ________ fraction.

Arranging in Order of Magnitude

Topic: Value of Decimal Fractions

Arrange the following in order of size. On the first line write the number worth most; on the last line write the number worth least.

1. ; .1 ; .99 ; .09 ; 9.9

_____; _____; _____; _____; _____.

Giving Approximate Answers

Topic: Approximations and Short Cuts in Multiplications

Below you will find a number of examples. Do not try to give the exact answers, but answers that are near to the exact ones. Use the short method of multiplying. If the multiplicand is 4207, and the multiplier ⅐, change the example to 4200 × ⅐. Work quickly. You will be allowed two minutes.

(*a*) 16⅔ × 3599 =
(*b*) 12½ × 2411=
(*c*) 33⅓ × 9010 =
(*d*) 25 × 8411 =
(*e*) 37½ × 2392 =

Indicating Intended Solutions

Topic: Two-Step Problems

To obtain the answer to each problem below you must perform two operations. Read each problem carefully and then decide what you will do first, and what second. Example: John started with ten dollars to buy some camping things. He bought a scout knife for $1.25; a fishing-rod and reel for $3.75; hooks and sinkers for $.65. What did he have left?

First: Add $1.25, $3.75, $.65
Second: Subtract total from $10.00

1. A tire was guaranteed for 10,000 miles. It was used as follows: In March, 800 miles; in April, 920 miles; in May, 1400 miles; in June, 1380 miles; in July, 2460 miles; in August, 2640 miles. On September 1 the tire was completely worn out. The owner makes a complaint to the man who sold him the tire. How many more miles was he justified in expecting his tire to last?

First I would ______________________________
Then I would ______________________________

2. Tires can be bought for $5.45 a piece. Mr. Jones decides to buy a complete set of these new tires for his automobile. He has $20. in cash. He knows that the storekeeper will trust him for the rest. How much will he owe the storekeeper?

First ______________________________
Second ______________________________

3. A real-estate dealer sold a plot of land at a loss of $4055. The day previous he sold six lots and a house, making a profit of $4200 on the lots and $3450 on the house. What was his full gain?

First ______________________________
Second ______________________________

Not until the reader has formulated a series of such tests will their advantage become clear. True, the making of new-type tests takes time, but some of this time may be made up by the ease with which they are scored. In the end, their final value must be gauged, not by time expended on them, but by the help they render both teacher and pupils.

Problems for Study and Discussion

1. Devise new-type tests in arithmetic, each designed to cover the work usually taught in a school month. For each test state the specific subject covered, and the specific objective or objectives. Be sure to follow types of questions presented in this chapter.

2. Devise two tests of the traditional type on the subjects used in formulating the tests in question 1 above.

3. Compare the two kinds of tests, new-type and traditional tests, with reference to (*a*) amount of ground covered, (*b*) the specific weakness each reveals, and (*c*) the probable value of each as a measure of what children really know.

CHAPTER XIII

PROBLEMS IN THE SUPERVISION OF ARITHMETIC

Significance of size of class. In an older day large classes were condemned because learning and size of the class were assumed to be in inverse relationship. It was taken for granted that the larger the class, the less the pupils learned, especially in a subject like arithmetic. The researches of a more recent generation lead to the conclusion now too easily accepted—within the bounds of reason the size of a class does not influence appreciably the effectiveness of learning. But we know that it is extremely difficult to measure or even gauge the effect of the size of a class because other conditions, some very significant, cannot be factored out. The teacher of the large class is often not the teacher of the small class. When classes in different school systems are compared we frequently lose sight of the difference in course of study, in time allotment, in standards of performance, and in character of supervision. Better teachers are usually assigned to larger classes, hence superior teaching ability may counteract the alleged retarding effect of a large number of pupils. While the small class gives opportunity for greater personal help, closer supervision, more diagnostic procedure, and better follow-up of each pupil, these promises are not fulfilled by the teacher of limited ability.

We may safely conclude that the competence of the teacher, the character of the supervision, the nature of the course of study, and the regularity with which diagnostic tests and remedial drills are administered, are decidedly more important factors, within reasonable limits, in determining the results of instruction than the mere size of the class.

Time of day for instruction in arithmetic. Many classes begin the day's work with arithmetic because of the general belief that the most difficult subject should come in the first period when the learning curve is at its best. The first period of the day is probably not the most favorable learning period for most children. The personal habits of a pupil, the regimen of his daily life, the concreteness with which arithmetic is taught him, the vitality of his teacher, the character of the exercises assigned him, and the social significance of the applications which he must make of his number experiences—these are all more significant factors than time of day when the subject is taught.

Effective teachers generally open the day "with work that will set a good standard of cheerful, efficient production"[1] and seek to complete the least interesting activities early in the morning. Because adequate experimental studies of the effect of the time of day on teaching results have not yet been made, our statements must of necessity be tentative, subject to change with the gathering of conclusive data.

Self-checking and self-correcting. No system of correcting compositions that fails to develop in pupils skill in self-correction is considered satisfactory. This standard should obtain in arithmetic also. Children must be required to check results, find their errors, and make corrections after comprehension of a process or fact is established. Systems of checking arithmetical processes are as important as the acquisition of the arithmetical processes themselves. In succeeding chapters we shall discuss the teaching of basic operations and modes of checking each.

Where practice books or pads are used each child should keep a daily record of his scores. Simple line graphs should chart for each pupil his daily, weekly, and monthly progress in arithmetic.

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[1] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), p. 227.

An example of such a graph, made day by day by a child, follows:

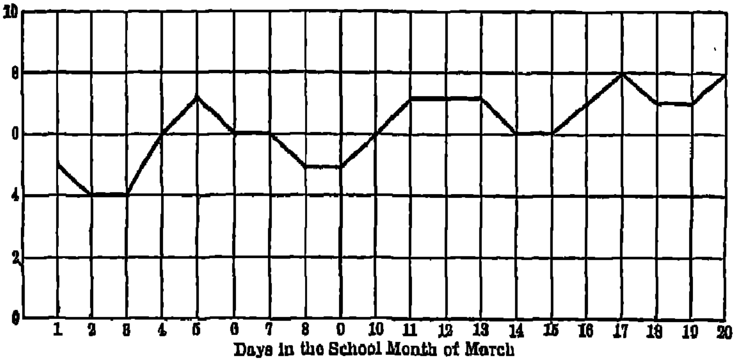

FIG. 4.—MY PROGRESS IN PRACTICE WORK IN MULTIPLICATION DAY BY DAY.

As far as possible, each pupil should be helped to make an analysis of his own shortcomings. "I generally get my 9 times wrong," concludes one child. "I get the wrong answer when I multiply by a number that has a zero in the middle, like *204*," observes another. "I get all my multiplication right but I do not put the numbers in the right place," says a third. Here we have the first step in improvement—recognition of a weakness in comprehension or manipulative skill.

When the child's mastery of fundamental combinations is weak the burden of checking may be excessive and the reliability of the child's procedure so poor that we may question the advisability of requiring such pupils to check their results—so the argument runs. Would not a like amount of time spent on additional examples be more profitably invested? We are inclined to argue for a decidedly negative answer to this question on at least two grounds: first, the habit of checking one's work is a method of work and like all methods of work, is most vital; second, the process of checking is as useful as the actual working of another example.

Reducing eyestrain in arithmetic. *Size of type.* Because of the limited field of vision of the eye, type that is too large requires too many eye fixations. Naturally it is helpful to give the eye, in one sweep, as much as the mind can reasonably grasp. The large type in the following example discourages the eye from encompassing 7, 3, and 5 in one glance; it probably sees only one figure at each glance.

547
383
555
638

Type that is too small involves eye strain and discourages attention to the figures in the example. The most satisfactory size of type for the several grades seems to be the following:

12 point type for Grades 3 and 4
11 point type for Grades 5 and 6
10 point type for Grades 7 and 8

The above example in addition, set in accordance with these standards of size, presents the following appearance:

Grades 3 and 4	Grades 5 and 6	Grades 7 and 8
547	547	547
383	383	383
555	555	555
638	638	638

Style of type. Distinctiveness is a vital consideration. The figures 5, 3, and 8 must be so clear that there is no danger of confusion. Fractional forms must not be too small. The horizontal line between numerator and denominator seems to be preferred, but, because of our present meagre knowledge of the hygiene of print in arithmetical forms, the slanting line cannot be condemned altogether. Since so much arithmetic is performed on paper, penmanship drills must include legible writing of figures as well as letters. Eyestrain caused by the reading of poorly made figures is probably greater than that caused by the reading of careless penmanship, because in the latter the context suggests the words even before the eye sees them.

Spacing. Proper vertical and horizontal spacing does much to give legibility to the page by helping to bring out the identity of each symbol. Too often the page of the textbook is crowded and does not invite close attention to the exercises.

The paper. It is important that the paper used for school books have sufficient body to prevent the print of one page from showing on the reverse side. The color should not be excessively white nor should the surface be so smooth as to afford the light a surface from which it is reflected.

The school can do much to bring about improvement in the format and print of books by demanding that they meet very definite physical standards. Undoubtedly commercial publishers would soon accept these physical requirements, if they had the sanction of authorities on the hygiene of the eye.

The copying of figures. A reasonable amount of time devoted to copying figures can readily be defended but some books entail serious loss of time by requiring excessive writing of figures before the child begins an arithmetical process. Practice books have an established place in the teaching of arithmetic, for aside from the quality of their content they offer many advantages. All of the child's time goes to arithmetical work and little or none to transcribing figures. The orderly, well spaced arrangement of the examples in the practice book serves as a model for imitation, especially when attention is called to these matters. When exigencies of class management give the child a little unexpected free time he has a ready means of occupying himself profitably. As the semester progresses each child has a cumulative record of his daily work. Practice books that are well graded and whose repetitions of number combinations occur with calculated frequency are really of inestimable value. A few of the types of exercises appearing in the better practice books are given below so that teachers may model their drill work after them.

1. Write the missing number. Follow example (*a*).

(*a*) $\frac{1}{8} = .12\frac{1}{2} = 12\frac{1}{2}\%$
(*b*) $\frac{3}{8} = .\underline{\quad\quad} = 37\frac{1}{2}\%$
(*c*) $\frac{5}{8} = .62\frac{1}{2} = \underline{\quad\quad}\%$
(*d*) $\quad = .87\frac{1}{2} = 87\frac{1}{2}\%$

2. Write the missing number. r. is remainder.

37 = $\underline{\ 6\ }$ 6's and $\underline{\ 1\ }$ r
45 = $\underline{\quad\quad}$ 6's and $\underline{\quad\quad}$ r
49 = $\underline{\quad\quad}$ 6's and $\underline{\quad\quad}$ r
53 = $\underline{\quad\quad}$ 6's and $\underline{\quad\quad}$ r

3. Write the whole number or the mixed number which each fraction equals.

$\frac{5}{4} = \qquad \frac{6}{6} = \qquad \frac{15}{5} =$
$\frac{9}{8} = \qquad \frac{11}{6} = \qquad \frac{12}{5} =$

4. Write the answers

8, 2's = 16
5, 3's =
6, 5's =
7, 4's =

5. Write the missing figure.

$\frac{4}{5} = \frac{8}{\quad} \qquad \frac{9}{8} = \frac{\quad}{8}$
$\frac{4}{7} = \frac{\quad}{14} \qquad \frac{9}{8} = \frac{\quad}{16}$
$\frac{2}{3} = \frac{6}{\quad} \qquad \frac{12}{9} = \frac{\quad}{3}$

6. Give the missing numerators or denominators.

$\frac{3}{4} = \frac{\quad}{8} = \frac{\quad}{16} = \frac{\quad}{24}$

$\frac{2}{3} = \frac{\quad}{6} = \frac{\quad}{9} = \frac{\quad}{12}$

$\frac{5}{8} = \frac{15}{\quad} = \frac{10}{\quad} = \frac{20}{\quad}$

7. Insert the two missing numbers in each example.

47 ÷ 6 = 7 and $\underline{\quad\quad}$ r. because 6 × 7 = 42 and $\underline{\quad\quad}$ = 47

39 ÷ 4 = $\underline{\quad\quad}$ and 3 r. because 4 × $\underline{\quad\quad}$ = 36 and 3 = 39

52 ÷ 7 = 7 and $\underline{\quad\quad}$ r. because 7 × 7 = 49 and $\underline{\quad\quad}$ = 52

8. At the end of a game of marbles,

A had 4 marbles
B had 6 marbles
C had 8 marbles
D had 10 marbles
E had 12 marbles
F had 15 marbles

a. Who has twice as many as A? .....

b. Who has half as many as E? .....

c. Who has three times as many as A? .....

d. Who has 1½ times as many as D? .....

e. Who has ⅔ as many as B? .....

Fill in the missing figures

f. E has ..... times as many as A.

g. A has ..... as many as E, etc.

To these illustrations we must add the more obvious exercises in the necessary arithmetical operations.

Group teaching in arithmetic. *Its place.* In our earlier discussions it was shown that arithmetic is a complex of skills; that there is no single general arithmetical ability; that proficiency in one branch of arithmetic offers no assurance of approximately equal proficiency in another branch of the subject. A small number of pupils may respond to a rational appeal when a new topic is presented; some require a very objective presentation; others can comprehend the reason governing the *modus operandi* only after using the new process again and again. Experimental evidence seems to show that native endowment is the most telling factor in determining proficiency in arithmetic. Mathematical judgment of younger children with less experience and training in arithmetic often surpasses that of older ones with three or four years' more study of the subject. The inference seems inevitable that arithmetic often requires group rather than class teaching and that it requires special forms of presentation and individualized drills to approximate the needs and the capacities of the homogeneous groups of pupils.

Devices for group teaching. We have already alluded to some of the more important devices for insuring group teaching in arithmetic. Some urge a *differentiated course of study* in which more intensive treatment or a greater number of topics is assigned to the bright children. While the course for the dull child is less extensive and less intensive it is not necessarily impoverished. What is relatively less necessary is omitted from the course for the second group of pupils. What to include and what to omit in the courses of study for the bright, the normal, and the dull are questions that have not yet been answered to the satisfaction of most educators.

Differentiated teaching, others urge, should be the keynote in group teaching. They argue: teach all approximately the same socially necessary experiences but by different methods. The able pupils can learn all that we demand of them through workbooks. The pupils of average ability should be taught by a developmental method. As the line of thought is unfolded by the teacher, they follow, step by step, and experience the thrill of thinking to a definite conclusion. A very objective process with much "showing" may prove very helpful to dull pupils. Hence the essence of the differentiated work lies in the differentiated methodology rather than in the differentiated content.

A third school plans its group teaching by devising diagnostic tests and remedial exercises. Each group has its differentiated drill, prompted not by the varying intelligence of each group, but by the varying needs of each group or even of each child.

A wise teacher does not align himself with any one of these schools but uses one or the other plan of differentiation, depending upon the kind of aid he wishes to give his pupils. A school with classes homogeneously grouped will still have need for group teaching, although the need is different from that felt when children are grouped by chronological age or by alphabet.

Difficulties in establishing group teaching. Group teaching is not popular because it requires arduous preparation by the teacher. "Excessive preparation" is the usual characterization. To devise profitable forms of practice which do not degenerate into mere busy work is no mean task. Teachers who are undaunted by these two conditions are reluctant to establish a plan of group teaching because written work must be checked. To supervise, inspect, and grade the work of each group is a difficult task. Not all teachers can control a class when different groups are engaged on different tasks, and all must be kept under direct supervision. These are not insurmountable obstacles but they are difficulties which discourage the timid.

Teaching with monitorial aid. A common classroom practice is the appointment of abler pupils as special coaches of their less proficient classmates. The benefits which the teaching pupil derives are social rather than mathematical. These are often counterbalanced by exaggerated estimates of ability and attending attitudes of superiority which are developed by these helpers. The backward pupil is not taught, but shown how; he is coached. We have conveniently assumed that this system of class coaching will lift the arithmetically submerged to the necessary level of proficiency. Nevertheless, we reject the conclusion that with slower pupils the teacher should not teach, but merely present each fact or process through purely mechanical imitation. The reader should not conclude that student coaches must not be appointed; the moral is rather that student aids must be prepared to render coaching service, but that not all types of guidance can be expected of them. They can be of inestimable help in checking written work and in the more mechanical phases of guidance.

Moot questions in the methodology of arithmetic. Arithmetic, like any other school subject, is replete with moot teaching questions. Too long have we sought final answers to many of them in contentious exchange of personal opinions. Because to-day, as never before, we understand that these academic debates must give way to facts derived from impersonal and scientific inquiry, there is feverish research activity in problems concerned with the teaching of arithmetic. So numerous are these researches that annual bibliographies and summaries of them are published by the University of Chicago.[3] Quantitatively the output is impressive but increased familiarity with it brings diminished respect. The investigations relating to arithmetic are uncoördinated studies carried out under conditions that predestine the findings to inconclusiveness. They arouse strong suspicion that their authors were more concerned with meeting the requirements for degrees or with maintaining professional demand for frequent publication than with finding truth.

Small wonder, therefore, that so much teaching of arithmetic clings to traditional practices and remains little influenced by these inquiries. We have no coördinated data to-day to answer some of the basic problems that arise in the teaching of arithmetic. Let us examine a few of these questions:

1. What shall constitute the content of the course of study in arithmetic? The answer must be formulated in terms of the arithmetical needs of intelligent adults in all vocations except those that are uncommon and highly technical.
2. Is there a general arithmetic ability? Are we to accept the conclusions that seem to be established by British investigators like Moore,[4] Green,[5] and Scott,[6] and by a few Americans like Collar,[7] or

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[3] See: "Summary of Educational Investigations Relating to Arithmetic," in University of Chicago *Supplementary Educational Monographs* published since 1925. The summaries are prepared by Guy Thomas Buswell.

[4] R. C. Moore, "The Psychology of Number: A Study of Numerical Ability," *Journal of Experimental Pedagogy*, Vol. 4, June, 1918, pp. 221–236.

[5] J. A. Green, "Normal Performances in Fundamental Arithmetical Processes," *Journal of Experimental Pedagogy*, Vol. 3, June, 1916, pp. 318–322; and Vol. 4, June, 1917, pp. 96–100.

[6] Colin A. Scott, "An Eighth-Grade Demonstration Class in the 3 R's," *Journal of Educational Psychology*, Vol. 10, April, 1919, pp. 189–218.

[7] D. J. Collar, "A Statistical Survey of Arithmetical Ability," *British Journal of Psychology*, Vol. 11, Part I, October, 1920, pp. 135–158.

the findings of Americans that seem to oppose a general arithmetic ability as expressed in the work of Stone,[8] Davies,[9] and Courtis?

3. Is it possible to eliminate most of the drill in arithmetic for the superior children? If so, at what point shall we draw the line between normal and superior children?

4. To what extent are our present textbooks in arithmetic in harmony with our knowledge of the psychology of arithmetic? By what standard can we assign to each commonly used textbook an index expressing its merit?

5. To what extent is motivation in arithmetic obvious? Does explicit motivation quicken the learning of arithmetic markedly? If so, to what extent?

6. To what extent can we diminish the place of arithmetic as an independent subject, and incorporate it in the day's manual and intellectual activities? Will such learning of arithmetic meet the demands of life adequately?

7. Should the same course of study be followed for all levels of intelligence? Wherein should they be differentiated? Should the differentiation be produced by eliminations or simplifications, or both? By what objective standard is the differentiation to be validated?

8. Should class teaching of arithmetic be supplanted altogether by a system of individual learning and individual practice like the Dalton Plan? If so, is this applicable to all types of children?

9. What standards of accuracy and speed may reasonably be set up in each fundamental operation for each school year in which the child learns arithmetic?

10. What loss, if any, would result from more incidental teaching of arithmetic in the first two years? Would the keener perception of the usefulness of arithmetic skill lead to greater application by the pupil in the third year and thus make up for the temporary loss?

11. What manipulative idiosyncrasies do the children exhibit in the processes of reducing certain techniques to habit?

12. What are the relative values of the most common objective aids used in the teaching of arithmetic?

13. Are drills based on multiple-sense appeal really helpful, or does the contribution of one sense set up interference with that of another?

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[8] Cliff W. Stone, *Arithmetical Abilities and Some Factors Determining Them*, Contributions to Education, No. 19 (Teachers College, Columbia University, 1908).

[9] G. R. Davies, "Elements of Arithmetical Ability," *Journal of Educational Psychology*, Vol. 5, March, 1914, pp. 131–140.

14. What correlation of efficiency exists among the several processes of arithmetic?

15. To what extent do mastery of language and ability to read enter into the efficiency which people develop in arithmetic?

16. What supervisory practices influence the learning of arithmetic?

17. What is the total technical vocabulary that arithmetic may introduce during the school course?

Our purpose is not to list all the investigations that must be undertaken in the teaching of arithmetic, but rather to indicate the character of the help that the practical but nevertheless professionally minded school man is expectantly awaiting. In the discussion of the teaching of specific facts and techniques we shall indicate, repeatedly, where decisions are still made on personal opinion because reliable data have not yet been obtained. What a tragic total we would attain were we to sum up all the research time that has been invested in idle inquiry! What, for example, can one do with the fact that addition is used twenty-four times as frequently as subtraction? Are we to content ourselves with a lower standard of accuracy in subtraction than in addition? Is subtraction less useful socially because it is employed less frequently? Why all the statistical travail to evolve an exact gradation of the additive combinations? What if 4+7 is slightly less difficult than 5+6? Are we to develop better control for the one than for the other? Will not some children remember 5+6 more readily than 4+7, or 4+7 more readily than 5+6? Should not drill beyond the initial stages, continuous drill, be determined by the needs of the individual child rather than by the graded difficulty of successive combinations? Gradation of number combinations that enables a teacher to arrange these combinations in four or five groups of progressive difficulty is extremely useful. A further refinement of gradation that assigns a place to each combination in a scale of difficulty seems futile. The introduction of the scientific method into educational practice holds out glowing hope. But, like all reforms, the movement has been done much harm by its extravagant devotees.

PROBLEMS FOR STUDY AND DISCUSSION

1. Plan an inquiry into the relation between size of class and proficiency of instruction. How would you organize the groups to be subjected to the experiment? What should be the procedure for each group? What uniform conditions would you maintain? How would you maintain these conditions? What test would you use at the end of the experimental period? Indicate clearly the "control" factor or factors.

2. "An effective system of correcting pupils' compositions has many characteristics of an effective system of correcting arithmetic papers." In discussion of this statement, include the characteristics referred to, and give an adequate illustration of each.

3. Examine at least two textbooks in arithmetic to determine the extent to which they meet hygienic standards. Tabulate the hygienic requirements of a textbook in arithmetic, and allot to each a numerical value according to its relative importance. The total of these values for a perfect book should not exceed 10. Score each of the two books according to your scale.

4. Consult the last six numbers of two or three professional journals such as *The Journal of Educational Research*, *The Journal of Educational Administration and Supervision*, *School Review*, *The Peabody Journal of Education*, *The Journal of Educational Psychology*, etc. Select three articles on some phase of the teaching of arithmetic. Prepare a short summary and a critical estimate of about 500 words of each article. Avoid such unsupported statements as, "This is a good article," or "This is an interesting study," or "I think the author has not proved his contention." Be sure to support your estimate of the author's work by specific references to his study.

CHAPTER XIV

MEASURING GROWTH IN ARITHMETIC

Educational results are gauged by personal opinion, by traditional types of examinations, and by standardized tests. The first two are the time-honored devices of teachers and supervisors; the third is a comparatively new instrument.

Limitations of personal opinion. As a standard of judgment, personal opinion is reliable in direct proportion to the intelligence of the judge, to the character of his training and experience, and to his ability to restrain personal interests from coloring his decisions. Prejudice, provincialism, denominational and sectional loyalties, inadequate experience—these are a few conditions that influence judgments. But personal opinion is still enthroned as the great judge in education. Matters as vital as the determination of the content of courses of study, the degree of skill to be acquired by pupils, and the methodology of teaching and learning are still fixed by personal opinion. Fetish after fetish is glorified and then discarded by the capriciousness of personal judgment.

The traditional examination. Educational growth is still too frequently measured by the fund of information which the child carries away. Written tests of information assume that prevailing curricula and methodology are essentially correct. But in the final analysis these tests gauge progress by what is temporary and relatively unimportant. Facts known to-day are forgotten to-morrow; mental habits—not facts—are the ultimate desiderata of instruction.

Let us consider the value of a test like the following, devised by teachers of a given grade, after they had taught (*a*) how to find areas and perimeters of rectangular figures, and (*b*) multiplication by fractions.

1. What is the area of a rug 9 by 12½ feet?
2. Which has a greater area, a rectangle 16 × 8 feet, or a square measuring 12 feet on a side?
3. Which (referring to problem 2, above) has the greater perimeter?
4. Find the cost of fencing a garden plot measuring 50 by 40 feet @ $1.50 a foot. A gate 4 feet wide will cost $24.

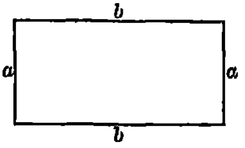
5–6. How many small school garden plots 20 sq. feet would you make out of a park plot 160 × 40 ft.?

Before answering the question, draw a plan to scale, showing how you would lay out the small plots so as to waste the least amount of planting ground.

7. Find the answer to the following:

a. 1⅔ × ⅞
b. 1⅔ × 1
c. 157 × ⅔
d. ¾ × 144
e. ⅞ × 0
f. ⅓ × ⅓

8. At $1.75 a yard what will 86 yards cost?
9. A gallon of a particular kind of oil weighs 8⅜ lbs. What will a barrel containing 54 gallons weigh?
10. Fill in the following blanks



a. *a* multiplied by *b* gives the ________ of the rectangle.
b. *a* + *b* + *a* + *b* gives the ________ of the rectangle.
c. (*a* + *a*) + (*b* + *b*) gives the ________ of the rectangle.
d. Side *b* is called the ________ of the rectangle.
e. Side (*a*) is called the ________ of the rectangle.

Such a test undoubtedly measures retention and comprehension; helps materially in determining a pupil's absolute standing and his relative rank in his group; encourages careful review of what was taught by insuring motivation; trains in ability to apply oneself to a given task; and indirectly, indicates the effectiveness of the teacher's skill in guiding his pupils.

Despite its usefulness such a test has marked limitations. It does not indicate the specific causes of a pupil's failures.

What if a child scores only 65%? The final grade does not reveal the specific difficulties which the child experiences. The relatively low grade may be due to one or more of the following conditions:

1. Failure to understand the condition, especially in problems 4 and 6;
2. Failure to understand the language, especially in question 10;
3. Failure to master the fundamental operations;
4. Carelessness in performing a process that is understood;
5. Inability to think in terms of quantitative relations.

This test is therefore not *diagnostic;* it is a measure of a composite of skills but not of specific or individual skills.

What if the class average is 58%? Is the teacher justified in assuming that the class did poor work? Or is the result reasonable for a given grade? There is no basis for a final answer. If this test had been given to thousands of school children, under uniform conditions, it would have been possible to formulate norms of achievement for different age and grade levels. The teacher would then have been able to determine how her pupils compare with others of similar age and grade. Therefore this test lacks *standardization.*

The standardized tests. The most accurate instruments yet devised for measuring instructional results are standardized tests in which the constituent tasks are carefully analyzed, tested, and assembled, so that the resulting test or examination is like a yardstick, every subdivision of which is of equal dimension. Such tests have at least four characteristics:

1. *They are standardized.* Every process that enters into the test is charted and its relative difficulty determined not by personal judgment but by results obtained in testing children. The values ascribed to each step in the test are determined with reasonable accuracy. Norms of achievement for children of different ages and grades are evolved from the data obtained by testing pupils under uniform conditions.

2. *They are objective.* At no point does personal judgment alone determine any of the quantitative aspects of the test.

The relative difficulty of each example and of each process and the relative value of each correct response are determined after careful statistical treatment of the responses of the many pupils tested. The plan of administering the test and of scoring the answers is prescribed to insure the degree of uniformity which makes results comparable. Nothing, or rather an absolute minimum, is left to the judgment of one person.

3. *They have validity.* The extent to which a test measures what it sets out to measure is its validity. A test in arithmetic may involve little arithmetic and much reading; it becomes therefore a reading test. A test in percentage may require little computation in terms of percent but much knowledge of business practice; therefore it lacks validity as a test in percentage. To be valid, a test must embody subject-matter that is considered socially necessary by the best professional opinions, textbooks, and courses of study; it must concern itself with the minimum essentials of the subject; and it must phrase its questions and problems in terms of a vocabulary that, by actual test, is not beyond the pupils. As a rule the composite judgment of recognized authorities is a method often used in studying the validity of a test.

4. *They have reliability.* The degree to which a test measures accurately and consistently that which it sets out to measure is its reliability. Repeated use of the same test at proper intervals, or of different forms of the same test, should yield results that are approximately equal. Differences in test scores so obtained are inevitable for many reasons: (*a*) Children differ in attitude and attention at the times when they take the same or different forms of the same test. (*b*) Physical conditions of the children or of the place where the test is taken influence the character of the response. (*c*) Taking the test is in itself educative for it gives practice in meeting a specific type of experience. (*d*) Children grow in knowledge and skill between test times. (*e*) The tests may have inherent structural weaknesses. In the development of a standardized test values must be reassigned, test elements must be changed,

and scoring techniques must be modified, until repeated use of the same test, after suitable intervals, yields approximately similar results for the same pupils. Before using a standardized test it is imperative that we ascertain its reliability.

Characteristic questions to be answered by scientific testing. We need accurate measures to determine the correctness of much that we now accept on faith or because of the force of mere verbalism. Among the many questions that need answers in terms of objective standards we may list the following:

What degree of accuracy and speed in arithmetic shall we expect in each of the successive grades of the elementary-school course? Each teacher gives assurance that she is developing habits of accurate and rapid manipulation. How are we to determine the degree of accuracy and speed which the children possess at the beginning and at the end of a grade?

How reasonable is a standard which a teacher or a supervisor sets up in arithmetic for a given grade? What if teacher and supervisor differ in their estimate of the ability of a pupil, or the average ability of the class? Whose judgment shall prevail? How shall the personal judgment be checked? What degree of skill should children usually attain at the end of a given grade?

Is it wise to eliminate all systematic teaching of number in the first year? In the first two years? Will children, beginning systematic work in the third grade, progress at an accelerated rate because of their maturity and attain the same degree of skill by the end of the sixth grade as their schoolmates who are given systematic number work from the earliest weeks in the first grade?

What is the relation between time devoted in arithmetic and the level of proficiency that children will attain in it? *A priori*, proficiency ought to be in direct proportion to the time spent. But investigations thus far do not warrant such a conclusion. Time expenditure is no guaranty of proficiency. We were told in an earlier chapter that if we were to select at random a school from a list of schools that give more than average time to arithmetic, the chances are about equal that we would have chosen a school in which the scholarship in arithmetic is below average; that if we were to choose at random a school from among those that devote less than average time to this subject, we would probably find the children doing superior work in it.

To what extent shall we rationalize in arithmetic? Shall children know the *why* for every step in multiplication by numbers of two or three orders, or shall they merely habituate the process? The advocates of rationalization are as insistent as their opponents, yet neither side has any more definite basis than personal opinion.

Shall children be taught to subtract by a method of addition or by a method of subtraction? The former method is now in the ascendency and is prescribed in many communities. Have we determined by reliable tests that children taught by one method do speedier and more accurate work than do those taught by another?

To what extent is arithmetic ability related to general ability? Shall we accept the explanation that arithmetic ability is so specialized that a child who does exceptionally good work in reading and in composition may lack that distinctive ability needed in arithmetic; that he may have a blind spot for arithmetic? Shall we give a more sympathetic ear to the explanation that such a child prefers language work to arithmetic, and therefore attends to the former and neglects the latter?

To these questions we must add the numerous unsettled problems presented throughout our study. They lead to the inescapable conclusion that too long have we sought answers to vital problems through personal opinion rather than by means of objective data scientifically derived.

THE DEVELOPMENT OF SCIENTIFIC MEASUREMENT IN ARITHMETIC

Scientific measurement of arithmetical ability, though comparatively new, has progressed so rapidly that it is now in its third stage of development.

The introductory stage, the non-standardized tests, 1900-1909. *The work of Rice.* The pioneer efforts to formulate scientific standards in arithmetic are generally credited to Rice. In 1902 he gave his test questions to about 6000 children of Grades 4A through 8B in seventeen schools representing seven cities. The test questions are given in the Appendix. The most striking results were the low ratings and the wide range of capabilities of children in the same grade. In the seventh year, the averages for pupils varied from 8.9 per cent to 81.1 per cent; in the eighth year, from 11.3 per cent to 91.7 per cent. The method of scoring was carefully planned to achieve uniformity and objectivity.

Rice then set himself the task of interpreting the results constructively. He began with the assumption that the probable factors determining successful work are home, the size of the class, age of pupils, time of day when a subject is taught, time devoted to the subject, home study, methods of teaching, teaching ability of the staff of instructors, the course of study, and supervision of teachers and standards for measuring results. Each of these factors was scrutinized in the light of the children's results.

1. The home is shown to be a negligible factor in determining quality of results. The findings showed poorer results in schools in good neighborhoods, where conditions were favorable to study.
2. The size of the class appears insignificant as a determining condition. Schools with the lowest averages had very small classes and some of the schools with the best results had very large classes.
3. The age of the pupil must be considered, Rice concluded, because the older pupils grasp the work more readily and the results are correspondingly better.
4. The schools examined in the morning did better than those tested in the afternoon, but the difference was slight, barely 3 per cent.
5. Those schools that gave most time to the subject were not correspondingly high in attainment, for the school that ranked

next to the lowest gave the greatest amount of time daily to arithmetic; the school that averaged 80 per cent and the one that scored only 25 per cent gave the same number of minutes daily.

6. The amount of home work given to the pupils seemed to bear no direct relation to the proficiency attained, and Rice concluded that home work puts an added burden on the children, saps their energies, and has practically no compensating results.

7, 8 and 9. The investigator's conclusions—that teachers whose work is modernized, who have teaching gifts and professional skill, and who follow a course of study designed to meet social needs do not influence the pupil's attainment—are difficult to accept. One is left with the suspicion that the author is more concerned with achieving triumph than truth.

10. Rice's final conclusions are that proper supervision of teachers and definitely standardized measures of results are the determining factors in making for efficient work in arithmetic. These standards of attainment do not permit each teacher to be the judge of his or her own work, nor each principal to be sole judge of the standing of his school or the worth of his teachers. Standard tests set up definite, well-crystallized expectations that give direction to each teacher's efforts.

Rice does not see, however, that these standards may become fixed and formal; that the routine test encourages routine work and discourages initiative. His work was too limited, and the absolutely uniform conditions necessary for determining a standard were not secured in each school tested. His contribution is significant, however, because of the impetus it gave the scientific movement in arithmetic.[1]

The work of Stone. The task which Stone set for himself was to secure data on (1) the nature of the results obtained by teaching arithmetic in the first six years, and (2) the relation of "distinctive procedure in arithmetic to resulting abilities." With these ends in view he organized a test of two parts: Part A, in fundamental operations; Part B, in reasoning problems. The former sought to combine all the difficulties in the fundamental operations, and the latter, to test the reasoning power of children in problems that deal with situations equally familiar to all and that omit mere memory requirements. The tests are given in the Appendix.

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[1] J. M. Rice, "Educational Research: A Test in Arithmetic," *Forum,* Vol. 34 (1902), pp. 281-297; "Educational Research: Causes of Success and Failure in Arithmetic," *Forum,* Vol. 34 (1903), pp. 437-452; *Scientific Management in Education* (Hinds, Noble and Eldredge, 1912).

The technique of the test was carefully worked out. Time limits, allocation of credits in scoring, gradation of tasks, and other elements vital in the formulation and administration of tests were determined, in the main, objectively. Subjective judgment in these matters was clearly held in check. In 1907 these tests were given to 6000 children in 152 classes in 26 school systems representing communities in the East and in the Middle West.

The following are the most important conclusions summed up by Stone as a result of his investigations:

1. Marked discrepancies exist among children of the same grade who are grouped on the basis of uniform teaching and approximately equal attainment.
2. Ability in one branch of arithmetic is no guaranty of approximately equal ability in another, even though the two abilities are both in fundamental operations. There is no such thing as a single or unified or combined ability in arithmetic. What we call ability in arithmetic is really a composite of abilities—a complex of abilities.
3. There seems to be no clearly defined relation between time devoted to arithmetic and proficiency developed in it. As time cost is increased, up to certain limits, children seem to do better work in fundamental operations than in reasoning problems. The probable conclusions are:
   a. Time given to fundamental operations is more productive of results, or,
   b. More of the total time of arithmetic should go to fundamental operations, or,
   c. Power to reason is less conditioned by amount of teaching than is skill in fundamental operations.
4. There seems to be no relation between the arithmetical abilities that children acquire and the quality of the course of study according to which they are taught. Stone submitted the twenty-six courses of study of those school systems whose children he tested to twenty-one competent judges who arranged courses in order of merit. When the children's ratings were compared with the course of study, no relationship could be

discerned. The chances seemed about even that the children taught from a course of study rated as low would attain higher scores, and conversely.

Stone's work showed clearly the possibilities of developing scientific standards for measuring the instructional results in arithmetic and for checking time-honored opinions of methods of teaching and of courses of study.[2]

The second stage, the introduction of standardized tests, 1909-1915. *The contribution by Courtis.* The most significant contribution to scientific measurement of the results of instruction was made by Courtis. His first arithmetic tests were not only standardized but also diagnostic in their implication. They measured, not the facts learned, but the habits developed in the process of learning—not the temporary, but the permanent acquisitions. The same test was given to all children in all grades under prescribed conditions that insured uniformity of administration and scoring. The measuring medium was kept constant.

The original tests came in eight parts, each seeking to measure one group of closely related skills. Tests 1-4, inclusive, sought to measure, in four one-minute tests, mastery of the combinations in each of the four fundamental operations. Test 5, a speed test in copying figures, measured motor ability and correctness of perception. Test 6, a speed test in reasoning, presented a number of problems and asked the children to indicate, but not to perform, the arithmetical operations that led to the correct answers. Test 7 was limited to fundamental operations that required paper-and-pencil computation. Test 8 measured ability to solve two-step problems.

A few years later Courtis devised supplementary tests known as Series B. These seek to measure power in addition, subtraction, multiplication, and division in actual examples. The test elements involve all the factors which enter into the four fundamental operations. These tests were planned to be given for the first time in the third year and then once or twice in each succeeding grade, so that supervisors might determine under what conditions children acquire the techniques that make for mastery of the fundamental operations.

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[2] C. W. Stone, *Arithmetical Abilities and Some Factors Determining Them*, Contributions to Education, No. 19 (Teachers College, Columbia University, 1908); *Standardized Reasoning Tests in Arithmetic and How to Utilize Them* (Teachers College, Columbia University, 1916).

The procedures followed in the development of the tests were refined sufficiently to produce a yardstick for measuring specific arithmetic abilities with unprecedented accuracy. The gradation of test elements, the allotment of credits to each, and the time limits were all determined as objectively as possible. The testing instrument which Courtis evolved consists of units of approximately equal measures of ability. Both the original tests and Series B are given in the Appendix.[3]

Extended application of the Courtis Tests revealed significant weaknesses: the problems were not practical enough, language difficulties were not factored out, so that it was impossible to ascertain whether a child's failure to obtain the correct answer was due to his limitations of language or of arithmetic. By actual test it was found that many satisfactory clerks obtained test scores that were lower than the norm of the eighth grade. Courtis was among the first to see these weaknesses in his tests.[4] Despite these limitations his contribution is most significant for the development of scientific measurement.

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[3] S. A. Courtis, "Measurement of Growth and Efficiency in Arithmetic," *Elementary School Teacher*, Vol. 10, pp. 58–74, 177–199; Vol. 11, pp. 171–185, 360–370, 528–539. In these articles the origin and development of the Courtis Tests are described.

———, "The Courtis Tests in Arithmetic," *Report on Educational Aspects of the Public-School System of the City of New York to the Committee on School Inquiry of the Board of Estimate and Apportionment*, Vol. I, Part II, Subdivision I, Section D (City of New York, 1913) pp. 390–546. Published later by World Book Co., Yonkers, N. Y.

———, *The Gary Public Schools: Measurements of Classroom Products* (New York General Education Board, 1919), pp. 146–215.

———, *Courtis Standard Research Tests: Third, Fourth, and Fifth Annual Accountings*, 1913–1916, Bulletin No. 4 (Detroit, Department of Coöperative Research, 1916).

[4] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), p. 48.

The third period: the refinement of standardized tests. After 1915 tests in arithmetic reveal an unmistakable diagnostic purpose, which they achieve in varying degrees. In all of them the research technique shows progressive refinement. The *Woody Arithmetic Scales*[5] were developed according to the difficulties experienced by children and not, as the Courtis tests, according to the logic of arithmetic. Woody judged difficulty by two factors: time needed to obtain the results, and the correctness of these results. The examples in the scales were selected not because they represent typical arithmetic situations, but rather because they brought consistent reactions by children. Each scale is made up of as large a variety of examples as any of the fundamental operations will permit. The first example is very simple, so simple that it is almost obvious; the last example is so difficult that a very small percentage of eighth-grade children can hope to solve it correctly. The examples in a scale are set in a series of progressive difficulties. The Woody Scales differ, therefore, from the Courtis tests in which the examples are of equal difficulty. Their diagnostic character with reference to class errors is the chief contribution of the Woody Scales. These scales are not without their limitations. The examples are chosen, not because of their social value, but because of the consistency of the reactions which they evoke in pupils; ability to perform any one example is inadequate evidence of mastery of the process involved. *The Cleveland Survey Arithmetic Tests*[6] are modifications of the Courtis Tests, and designed for Grades 3–6. They measure skill in fundamental operations by carefully graded

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[5] Clifford Woody, *The Woody Arithmetic Scales: How to Use Them* (Teachers College, Columbia University, 1920); *Measurements of Some Achievements in Arithmetic,* Contributions to Education, No. 80 (Teachers College, Columbia University, 1920).

[6] Charles H. Judd, *Measuring the Work of the Public Schools* (Cleveland, Ohio, Survey Committee of the Cleveland Foundation, 1916), pp. 94–123. Gives a full description of these tests and the results of their administration in Cleveland.

ARITHMETIC TESTS OFTEN USED TO DETERMINE PROFICIENCY IN NUMBER WORK

THE TEST	THE AUTHOR	MEASURES F (FUNDAMENTAL OPERATIONS) OR P (PROBLEM-SOLVING)	GRADES	PUBLISHER	REMARKS
Diagnostic Chart for Individual Difficulties in the Fundamental Processes of Arithmetic	Buswell-John	F	3-8	*	An individual test
Scale for Problems in Arithmetic	Buckingham	P	3-8	*	Diagnostic and designed to lead directly to remedial activities in 85 unit skills. (N.B. This applies to Compass Test)
General Survey Arithmetic Test	Monroe	F	3-8	*	
Standardized Reasoning Test in Arithmetic	Monroe	P	4-8	*	
Diagnostic Tests in Arithmetic	Monroe	F	4-8	*	Diagnostic function is distinctive
Wisconsin Inventory Test in Arithmetic	Osburn	F	2-8	*	
Arithmetic Reasoning Test	Otis	P	4-12	†	205a
Mixed Fundamentals	Woody-McCall	F	3-8	‡	Modification of the Woody Arithmetic Scales
Arithmetic Reading Test	Stevenson	P	4-9	‡	
Metropolitan Achievement Tests	Bayne, Graham, and Orleans	F and P	3-8	†	Group survey test; gives diagnostic aid
Stanford Achievement Tests	Kelly, Ruch, and Terman	F and P	3-8	†	Survey test; indirectly diagnostic
Sangren-Reidy Instructional Tests	Sangren and Reidy	F and P	2-8	*	Group test distinctly diagnostic

* The Public School Publishing Co., Bloomington, Ill.

† World Book Co., Yonkers, N. Y.

‡ Teachers College Bureau of Publications, Columbia University, N. Y.

examples. *The Stanford Achievement Tests*[7] and the *Metropolitan Achievement Tests*[8] are typical of the tests of this period. In the table on page 172 are listed the arithmetic tests frequently employed to measure proficiency in number work.

Diagnostic vs. survey tests. *Three types of tests.* The inventory test is given before systematic instruction to determine precisely what knowledge and what skills the pupils possess. Such an inventory test, by taking stock of what pupils already know, indicates to the teacher what may be assumed and where the teaching of new experiences must begin. The norms of appropriate standardized tests enable the teacher to compare her pupils with others of similar age and grade.

During a lesson or in the course of a series of lessons on a given topic it becomes necessary to check children's comprehension and to gauge the degree of skill which they have acquired. On such occasions, teacher-made tests, oral or written, are given. These tests are usually short and consist of questions designed to give the teacher the specific information desired before teaching can proceed economically. Thus, the teacher is about to present cancellation when the common divisor is not actually seen in any numerator or denominator, for example, $\frac{4}{9}\times\frac{6}{7}$. Six examples like the following are presented for rapid solution: $\frac{3}{4}\times\frac{5}{3}$ and $\frac{3}{4}\times\frac{1}{6}$—in which the common divisor is clearly seen in the examples themselves. If the correct answers are promptly given and if the children can justify the process of cancelling in terms of "dividing numerator and denominator by the same number," then clearly they are ready for the new type, $\frac{4}{9}\times\frac{6}{7}$, in which the common divisor, 3, must be found by comparing the two multiples, 6 and 9.

A third test, administered after teaching, is designed to serve as a diagnostic instrument to reveal the common errors made in a class and to give indication of the probable causes. Diagnostic tests are regarded by many teachers as the most useful of all tests. The most helpful diagnostic tests are usually those prepared by teachers to meet the specific needs of a given class. Let us assume that children have been taught all the multiplication combinations with products not in excess of 36. They are expected, therefore, to know 4×9, 3×6, 5×7, 3×12—all the factors of any multiple not greater than 36. To devise a diagnostic test the teacher lists all the combinations for ready reference. A stencil is then ruled to show four columns and each combination is introduced four times but not in succession.

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[7] Kelly, Ruch, and Terman, *Stanford Achievement Tests* (World Book Co.).

[8] Bayne, Graham, and Orleans, *Metropolitan Achievement Tests* (World Book Co.).

At the signal begin to answer the following questions. When you hear the word stop, put down your pencil at once. Do as many as you can and try to answer all.

5 × 6 = ____
9 × 4 = ____
4 × 6 = ____

6 × ____ = 24
6 × 5 = ____
36 = 4 × ____

9 × ____ = 36
____ × 5 = 30
24 = 4 × ____

6 × 4 = ____
4 × 9 = ____
30 = 6 × ____

Notice that the combination 9×4 appears in each column but in different form. This is true also of 5×6 and 4×6. There is no regularity in placing each of the four forms of a combination. A child who is correct in all the four forms of 9×4, or in three of them, may be assumed to know this combination.

After such a test has been corrected, and a tally made of the number of times each item was incorrect, the teacher knows:

a. Items which have been missed by an appreciable group and therefore merit the attention of the whole class
b. Items which are known by the group as a whole
c. Combinations which should be practiced by individual children

Individualized and differentiated drill, not uniform drill, should occupy most of the time devoted to habituating number combinations. Without a diagnostic test there is no adequate basis for this kind of differentiation and individualization.

Let us analyze the plan of another diagnostic test made by a teacher. It includes six types of examples in subtraction of common fractions, each repeated two or three times.

PLAN OF A TEACHER-MADE DIAGNOSTIC TEST

The Type	What Is Tested	Number of Times Repeated
1. $5\frac{5}{8}-1\frac{3}{8}$	Whether children will recognize the common denominators and subtract without borrowing. Some children say $\frac{3}{8}$ from $\frac{13}{8}$ even though there is no need of borrowing.	2
2. $5\frac{5}{8}-1\frac{1}{4}$	The object is the same as in type 1, except that there is need of reducing the one quarter to $\frac{2}{8}$.	2
3. $5-\frac{7}{8}$	Whether children will convert 1 unit into $\frac{8}{8}$ without adding any fraction.	2
4. $5\frac{1}{8}-1\frac{7}{8}$	Whether children can borrow without changing to common denominators.	3
5. $5\frac{1}{8}-1\frac{3}{4}$	Whether children can borrow and change to a common denominator.	3
6. $5\frac{1}{8}-1\frac{2}{3}$	The object is the same as in type 5 except that the common denominator is not one of the denominators in the example.	3

After such a diagnostic test is scored and the errors tallied the teacher knows what to reteach, which pupils need no further practice, which pupils need further drill, and, finally, the content of the drill material for different groups which have not met the required level of proficiency.

A tabular contrast of survey and diagnostic tests may fix the distinctive character and functions of each.

<table>
<tr><th></th><th>The Survey</th><th>The Diagnostic Test</th></tr>
<tr><td>1.
The Scope</td><td>1. Usually extensive enough to include all the essentials in a large division of the subject, for example, in fractions, in decimals, in fundamental operations of whole numbers, in two-step problems.</td><td>1. Generally of limited range because the aim is to cover all types of difficulty in one limited process or in one field of subject-matter.</td></tr>
<tr><td>2.
Organization</td><td>2. The test elements are grouped into sections. Illustration: Part I, addition of fractions; Part II, subtraction of fractions; Part III, multiplication of fractions; etc.</td><td>2. The test consists of types, each of which is designed to reveal an element of strength or weakness. Example: see test above on subtraction of common fractions.</td></tr>
<tr><td>3.
Time</td><td>3. Time not a vital factor because the test must be long enough to cover its field.
It is best to follow the time set by the maker of the test. In general, it may be said that speed is overstressed in survey tests.
These tests are given at the beginning or the end of a semester.</td><td>3. The limited nature of the field makes possible a shorter test. It is wise practice to give many short diagnostic tests.
The child should be allowed all the time he needs. Time is significant only as it is an index of need; thus, if the child is inordinately slow, he needs stimulation for more rapid work.
Diagnostic tests are given at the end of a series of lessons devoted to a unit of subject-matter and its accompanying skills.</td></tr>
<tr><td>4.
Validity</td><td>4. Determined by the choice of the test elements. A sample of each type of example usually prescribed in approved books or courses of study is included.</td><td>4. Produced by good samplings of difficulties in one process. Validity is usually achieved by having a large number of examples in a narrow field.</td></tr>
<tr><td rowspan="2">5.
Norms and Standards</td><td colspan="2">A norm represents the achievement of the average pupil or the average performance of a group of pupils. A standard is higher than a norm and refers to a level of achievement beyond which it is usually uneconomical to continue to practice.</td></tr>
<tr><td>5. A norm in a survey test is valuable for general comparisons but not for absolute comparisons because we neglect the native differences in the two groups, the extent of preparation of each group, etc. A norm here is a quantitative measure of performance.</td><td>5. A norm is almost useless in diagnostic testing and teaching because interest is focused on learning techniques and special remedial practices. More than a norm we need a list of the most common errors made by learners and the frequency of each error. To illustrate:
Difficulties in Multiplication of Decimals
Misplacement of Decimal Point 40%
Errors in multiplication (facts) 24%
Omission of decimal point, etc.* 7%</td></tr>
</table>

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* For full and accurate list of errors in multiplication of decimal, see, L. J. Brueckner, "Analysis of Difficulties in Decimals," *Elementary School Journal*, Vol. 20, September, 1928.

Individual diagnoses. Errors may be grouped into (1) class or group errors and (2) individual errors made by comparatively few pupils. A diagnostic group test is designed to reveal the former. Analysis of individual work may be the only way of discovering the latter.

Children exhibited a great variety of methods of working in arithmetic. Some of these were undoubtedly taught by teachers, others were acquired in unsupervised study. Sometimes methods taught at home are in sharp conflict with the procedures of the class. Laymen frequently fail to understand that even though a method leads to the correct answer it may nevertheless be undesirable. On occasions school people have uncritically adopted the standard of the business world—correct answers regardless of the method of attaining them. Counting on fingers, excessive talk in column addition, resolving $9+6$ into, $9+1=10$; $10+5=15$, instead of instantaneously recognizing the pattern, $9+6=15$—these are illustrations of unwise procedures that lead to correct answers, but which inculcate very deleterious habits of work. By questioning children and studying their modes of work we may discover individual weaknesses that require prompt correction.

A preliminary diagnostic test may be given to a whole class. This is followed by conferences with individual pupils to uncover full extent of personal weaknesses and to prescribe remedial measures. Careful questioning and observation of a pupil's methods of work may reveal one or more of the following weaknesses in subtraction of whole numbers:

1. Inadequate mastery of combinations, $17-8=7$.
2. Unfamiliarity with borrowing process, $\begin{array}{r}52\\ \underline{17}\\ 40\end{array}$; 7 from 2 presents a difficulty which is resolved by placing a 0 in the answer.
3. Neglect to complete the borrowing process, $\begin{array}{r}52\\ \underline{17}\\ 45\end{array}$. The pupil performs the first step correctly but does not carry the process through by taking 2 from 5 or 1 from 4.
4. Inability to manipulate 0 in the minuend $507-243$.

5. Minuend is subtracted from subtrahend numbers to avoid borrowing, $\begin{array}{r}724\\ \underline{236}\\ 512\end{array}$. Since 6 from 4 causes trouble, the child takes 4 from 6.
6. Carelessness.
   a. Adding instead of subtracting.
   b. Omitting a column or a number.
   c. Misreading or miscopying numbers.
   d. Confusion of column by subtracting the tens from the hundreds, $864-231$; 1 from $6=5$.
7. Bad habits of work.
   a. Counting on fingers to obtain $9-7$.
   b. Talking in written work; $52-17$, "7 from 2 I cannot take, so I __________," etc.
   c. Deriving unknown facts from known, for example, $45-6$; $15-6=?$ $15-5=10$ but 5 is one less than 6 hence the answer is 9 and 39.[9]

If these errors and their causes are not revealed to the child they persist and lead to more errors in other fundamental operations. Such detailed diagnostic work helps the teacher to forestall errors, and thus reduces the extent of the diagnostic and corrective work which must be done later. But too few teachers know the common errors, their causes, and their relative rate of frequency in most arithmetic processes.

Objection is made that diagnostic analyses require considerable expenditure of teachers' time. That time is consumed is true beyond question. But we are likely to exaggerate the amount of added time that such diagnostic analyses will require. Not all children need such detailed diagnoses. Time is spent, but in the end time is saved by eliminating effectively the causes of persistent errors. And we must not forget that such sympathetic diagnoses save the child from the crushing sense of defeat that results from repeated errors.

Values of standardized tests. The wise use of standardized tests promises much to teachers. Those tests that are diagnostic aid in uncovering the specific difficulties which children experience. Help can now be given effectively through direct explanations and through practice material that is differentiated and individualized. The tedium of this type of drill is reduced considerably, because the pupil realizes his need for this specific corrective exercise. Proficient children are saved from much useless overlearning. Classroom teachers do not need more time for more drill, but they do need aid in formulating differentiated drill material to overcome the specific weakness of different groups of children. We have offered many illustrations to show that standardized diagnostic tests help to discover individual differences, and by implication, at least, indicate differentiated programs of remedial activities.

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[9] Similar analyses for addition, multiplication, and division are given in the treatment of these processes.

Supervision that is understanding and constructive is promoted by the administration of critically selected tests. They supply norms which indicate what may reasonably be expected of children of different levels of intelligence and in successive grades. Teachers and supervisors may set up personal standards for the various grades that are either too high or too low for their pupils. In the former case they harass their pupils by expecting an unreasonably high level of attainment. In the latter case they unwittingly teach their pupils how to get by without the expenditure of a full measure of effort. The achievement that may be expected of children of a given mental age and school grade must be determined by the average attainment of a comparable group. It cannot be formulated subjectively by adults.

For various reasons an accurate measure of a pupil's ability is frequently necessary. The judgment of the teacher is very inexact. Even in rating answers to questions in such subjects as elementary mathematics, experienced high-school teachers of geometry have ascribed to the same examination paper, values that vary from 20 to 90 per cent. Standardized tests prescribe methods of scoring that are uniform and minimize the capriciousness of individual judgment.

Standardized tests have helped to grade pupils in skill subjects. They have shown that in a heterogeneously organized class there is a greater divergence than similarity of ability; that 80 per cent of such a class may be replaced by children of other grades without changing the total scores; that a person employing a school graduate of average ability secures the services of an adolescent whose work can be equaled by 5 per cent of fourth-grade pupils. Such startling inequalities, exposed by standardized tests administered to all grades from the third to the ninth or tenth, have quickened the development of homogeneous grouping with respect to intelligence or specific abilities in a given subject.

When the effects of different courses of study, of different teachers, of different teaching techniques, or of different time allotments must be determined, standardized tests are of inestimable value. We have referred frequently to the futile attempts to settle educational matters by debate in which personal opinion is pitted against personal opinion. Standardized tests give us a relatively accurate yardstick for measuring the instructional results of a course of study or of teaching practices.

Limitations of standardized tests. Despite these values standardized tests are not without their limitations. Being derived from existing courses of study they tend to perpetuate traditional curriculum material. The norms offered by any standardized test are significant only as the tests measure mastery of knowledge and skills that meet the demands of actual life. Too often the tests perpetuate the kind of arithmetic that should be eliminated from class work.

But it may be argued that the purpose of a standardized test is not to determine the appropriateness of the material in a course of study, but rather to measure accurately the extent to which the prescribed subject-matter has been mastered.

Other objections to the use of standardized tests include such strictures as the following: the stress on these tests by supervisors may stultify class teaching by promoting coaching for periodic examinations; some tests have only limited reliability and validity; speed is too frequently overemphasized; teachers may limit their teaching to the type of examples that make up a test; the norm for a given grade may tend to become the maximum attainment for that grade and thus discourage children and teacher from attaining higher levels of proficiency. These objections indicate the unwise use of standardized tests—not limitations that are inherent in them.

PROBLEMS FOR STUDY AND DISCUSSION

1. Standardized tests have developed, in the main, because of acute dissatisfaction with personal opinion and teacher-made tests as measures of ability. Plan a careful discussion of this statement. Include as many illustrations as possible.

2. Explain and illustrate the following terms and expressions: *standardized test, the objectivity of a test, validity, reliability, inventory* or *diagnostic test, survey test, norm, standard.*

3. Consider the moot questions listed in the preceding chapter. Which of these can probably be settled with the aid of standardized tests?

4. Secure copies of the last three tests in the table on page 172 and their accompanying manuals of directions. Read these carefully. Try to secure permission to administer and score one of these tests. List the uses that can be made of the data thus obtained.

5. Assume that you have taught a class division by common fractions. Devise a diagnostic test on this process. Justify the organization of the test and the inclusion of each element in it.

6. With which of the values of standardized survey tests discussed in this chapter do you agree? With which do you take issue? Why?

7. We cannot repeat too frequently nor too emphatically that the primary purpose of a test is to indicate more effective means of helping pupils to grow. Illustrate the truth of this statement by referring to the three tests which you examined critically.

8. Express in a carefully planned statement your judgment of the following practice of a principal: At the end of each semester, a standardized survey test in excellent standing is administered to Grades 3–6. The term grade for each pupil and the estimate of the teacher's ability in arithmetic are determined by the pupils' scores.

9. Justify your agreement and disagreement with each of the following:

(*a*) "Speed is important in taking a *skill test*, but is relatively unimportant in a *power test*" (power as in solving problems).

(*b*) Cramming that follows honest day-by-day study may be justified.

(*c*) A diagnostic test teaches the folly of cheating on a test because such cheating is self-deception.

(*d*) A test should come at the end of each school month or semester.

PART V

LESSON PATTERNS IN THE TEACHING OF ARITHMETIC

CHAPTER XV

THE FORMAL LESSONS: INDUCTIVE AND DEDUCTIVE TEACHING

There are as many forms of lessons or recitations as there are modes of thinking. In general, the closer the teacher follows the natural mental processes of the pupil, the more effective is the teaching, for teaching is fundamentally the process which facilitates learning. The lesson plans which we shall discuss, therefore, reproduce the steps which the mind, left to itself, would follow in mastering specific experiences. We shall analyze the following recitations:

I. The formal lessons: inductive and deductive teaching.
II. The lessons for retention and recall: drill and review.
III. The lessons for power: study lessons, problem and project teaching.

Each of these modes of quickening the learning process has its distinctive place in the general scheme of teaching. None of them is always appropriate. An eclectic method formed by a combination of certain elements of each of these is not always possible nor helpful. The specific characteristics of any one of these lesson forms often make it either the most or the least effective mode of teaching a specific subject in arithmetic. We must stop, therefore, for a critical analysis of each of these lesson or recitation patterns.

The characteristics of induction and deduction. We assume that the student of special methods of teaching arithmetic knows that induction is that form of reasoning in which a general law is derived from a study of particular objects or specific processes; that the five formal steps of the Herbartian recitation or the Method-Whole follow a severely inductive development; that induction is the method which leads the pupil to "discover" knowledge; that the law, rule, or definition formulated by the child in inductive lessons is the summation of all the child has gathered in the presentation; that induction demands active coöperation on the part of the pupils throughout the lesson. The reader will find many illustrations of inductive teaching in the chapters that deal with fractions, decimals, percentage, and mensuration.

Strictly speaking, pure induction is seldom found in teaching. In all induction the generalization that is evolved is regarded as a tentative conclusion, a temporary hypothesis, held in abeyance until prolonged applications lead either to the establishment of its validity or to the modification of its content. The law obtained in an arithmetic lesson is usually final because the teacher selected only those individual instances which point unmistakably to it.

At times the law is evolved from a single typical instance, and not from a study of a number of specific individual cases. Thus, the general procedure for multiplying by any number of two orders, for example, by 18, is derived from the procedure followed in multiplying by any other number of two orders; or the rule for division by a decimal may be deduced from the specific example which requires division by *one* decimal fraction, for instance, by .6 or .06.

Characteristics of deduction. It is evident that deductive teaching, which begins by positing the law and then seeks to read meaning into it by applying it to a variety of specific examples, differs from induction in both procedure and spirit. In deduction the child does not discover laws, but develops skill in applying them. The lesson in division given in Chapter XXIII is an illustration of a typical deductive lesson, and shows that deductive teaching is not necessarily authoritative instruction imposed on the child without appeal to thought or motive.

Place of deduction in arithmetic. There are conditions in arithmetic when deductive teaching is not only advisable, but almost imperative. Chief among these we must include the following:

1. *When the generalization is too difficult.* There are laws in arithmetic that require powers of analysis and synthesis beyond the abilities of young children. An inductive lesson will produce confusion and discourage the child. For this reason we must teach deductively such topics as division by a number of two orders, multiplication by a number of three orders, and subtraction with numbers in the subtrahend larger than numbers in the corresponding places of the minuend.

2. *When the generalization is too long.* Children can derive an arithmetic rule inductively only when the individual cases presented are simple enough to enable them to make comparisons, to abstract underlying likenesses and characteristics, and then to weld these into a synthetic concept. Obviously we court confusion when we attempt to develop inductively the method of finding the least common denominator, or the method of dividing by a number of two or more places. The elements in these arithmetic processes are so numerous that no child can maintain enough control over them to enable him to formulate the generalization. Experience teaches that in such situations a deductive lesson is wiser, because it makes for economy of time and clarity of comprehension.

3. *When legitimately pressed for time.* Let us assume that a teacher of a given grade must teach division by decimals. A preliminary inventory test reveals a surprising lack of mastery of fundamental concepts concerning decimal notation—a topic prescribed by the course of study for an earlier grade. In making up the work of the lower grade the deductive teaching procedure is helpful and time-saving.

4. *When prescribed topics have little social value.* A professionally minded teacher is constantly evaluating the materials in the course of study in terms of social use. What a course of study demands, the teacher must, of course, carry out faithfully. But the teaching technique must be left to her. When in the judgment of the teacher a topic assigned by the

course of study lacks social value, she must feel entirely free to teach it by the most time-saving procedure at her command.

An illustration may avoid misunderstanding. A teacher of an eighth-year class in one of our city schools finds that she must teach the area of a circle to a class of girls whose education, at best, will not extend beyond the elementary school. It is evident that the formula, $A = \pi r^2$, will contribute no material aid in the adjustments which these girls must make immediately after graduation. To teach the area of a circle inductively will take an entire period and leave no time for application; to teach it deductively requires about one-fifth of the period and leaves considerable time for application of this formula and for practice of more vital arithmetical techniques. It is futile to argue that no course of study should be so inflexible. The truth is that courses of study are often inflexible and that the teacher is confronted by a fact, not a theory—she is expected to carry out the prescriptions of the course of study.

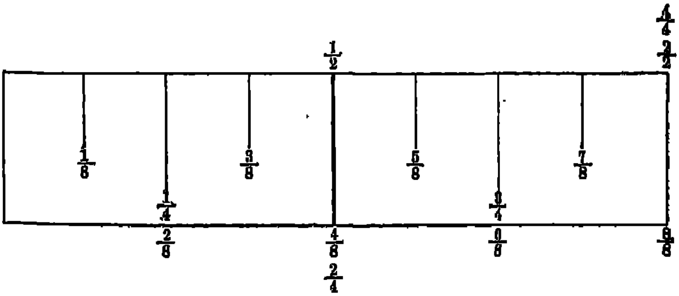
5. *In textbook teaching of arithmetic.* When children are taught how to study arithmetic they must of necessity learn from the printed page. Although the study techniques may be developed inductively the subject-matter of arithmetic is naturally studied deductively. A more detailed consideration of the study lesson with appropriate illustrations will be found in a later section of the book.

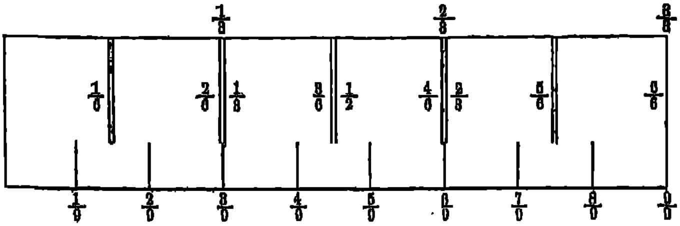
Comparative values of inductive and deductive teaching. Since the days of the Herbartians, inductive teaching has been enthroned. The time has come to challenge its sway. We have no real data, experimentally derived, to prove that deductive teaching gives pupils less control of content and fewer skills than inductive teaching. Educators have been successful in making out a seemingly logical case for induction. But in the final analysis, inductive teaching developed its authority through sheer verbalism. Its place in the scheme of teaching may be more precarious than we suspect.

The organization of the inductive lesson. The traditional recitation with its five formal steps, *preparation, presentation, comparison, generalization,* and *application,* still serves as a basis for many effective lessons in arithmetic. A first lesson on the reduction of a common fraction to lowest terms, in which children are taught to divide numerator and denominator by the same number, illustrates this form of recitation.

Preparation: Elicit that 50 cents and 5 dimes are the same; that 50 cents is $^{50}/_{100}$ of a dollar and 5 dimes is $^{5}/_{10}$ of a dollar; hence $\$^{50}/_{100} = \$^{5}/_{10}$; in similar manner, using 90 cents, $\$^{90}/_{100} = \$^{9}/_{10}$. Lead children to conclude that since the values are the same, it would be advisable to use smaller fractional forms in all solutions. Question children on what the next lesson in arithmetic should be. The answer, "How to change large fractions into smaller ones," is readily obtained. Note that the lesson has an *aim*, a *motive*, and an *apperceptive basis* or review of necessary old knowledge.

Presentation: Diagrams like the following prepared carefully on charts are then shown.





The effectiveness of these diagrams can be enhanced by using a variety of colors to indicate the different fractional parts, and by having children fold paper to reproduce these subdivisions.

With the aid of the diagrams elicit from the children the following equations:

$$\frac{2}{4}=\frac{1}{2} \qquad \frac{2}{6}=\frac{1}{3}$$
$$\frac{4}{8}=\frac{2}{4} \qquad \frac{4}{6}=\frac{2}{3}$$
$$\frac{4}{8}=\frac{1}{2} \qquad \frac{3}{6}=\frac{1}{2}$$
$$\frac{6}{8}=\frac{3}{4} \qquad \frac{3}{9}=\frac{1}{3}$$
$$\frac{6}{9}=\frac{2}{3}$$

Take the first fractional equivalents, $\frac{2}{4}=\frac{1}{2}$, and submit $\frac{2}{4}$ to an analysis like the following: by what must I divide the 2 to obtain 1; the 4, to obtain 2? The third, $\frac{4}{8}=\frac{1}{2}$; by what must I divide the 4 to obtain 1; the 8, to obtain 2? In the same manner analyze each equation until the blackboard work takes on the following appearance:

$$\frac{2\div 2}{4\div 2}=\frac{1}{2} \qquad \frac{2\div 2}{6\div 2}=\frac{1}{3}$$
$$\frac{4\div 2}{8\div 2}\quad\frac{2}{4} \qquad \frac{4\div 2}{6\div 2}=\frac{2}{3}$$
$$\frac{4\div 4}{8\div 4}=\frac{1}{2} \qquad \frac{3\div 3}{6\div 3}=\frac{1}{2}$$
$$\frac{6\div 2}{8\div 2}=\frac{3}{4} \qquad \frac{3\div 3}{9\div 3}=\frac{1}{3}$$
$$\frac{6\div 3}{9\div 3}=\frac{2}{3}$$

Comparison: Through questions elicit from the children that the same number is used to divide both numerator and denominator. The blackboard work is more effective if the common divisors are written in colored chalk.

Generalization: Children now attempt to formulate the law. The teacher makes the necessary modifications in their language.

Application: Many types of examples are now given for oral and for written solution, until the process is reduced to habit. To illustrate:

1. Reduce to lowest terms: $\frac{3}{6}$, $\frac{3}{12}$, $\frac{3}{9}$, $\frac{4}{8}$, $\frac{4}{12}$. Note that the common divisor is the numerator in each case.
2. Reduce to lowest terms: $\frac{6}{9}$, $\frac{8}{12}$, $\frac{15}{18}$, $\frac{14}{21}$. Why are these fractions more difficult than those above?

3. Which are correct statements?

a. $\frac{4}{6}$ is the same as $\frac{2}{3}$	*e.* $\frac{3}{4}$ is the same as $\frac{6}{8}$
b. $\frac{8}{12}$ is the same as $\frac{2}{3}$	*f.* $\frac{3}{4}$ is the same as $\frac{10}{12}$
c. $\frac{6}{10}$ is the same as $\frac{2}{3}$	*g.* $\frac{3}{4}$ is the same as $\frac{4}{6}$
d. $\frac{10}{15}$ is the same as $\frac{2}{3}$	*h.* $\frac{3}{4}$ is the same as $\frac{9}{12}$

4. Fold paper or make drawings with ruler and pencil to prove each of the following:

a. $\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$
c. $\frac{2}{3} = \frac{4}{6}$
b. $\frac{6}{8} = \frac{3}{4}$
d. $\frac{6}{12} = \frac{3}{6} = \frac{1}{2}$

5. Fill in the missing term in each of the following fractions:

$$(a)\ \frac{4}{6} = \frac{2}{-};\quad (b)\ \frac{6}{8} = \frac{-}{4};\quad (c)\ \frac{-}{6} = \frac{10}{12};\quad (d)\ \frac{6}{-} = \frac{2}{3}$$

If the reader will follow the lesson without noting that it was planned according to the Herbartian steps, the entire development will seem natural. The formal steps tend to put the burden of the work on the children, to lead them to discover the various relationships among numbers and the final law involved, to give them opportunity for application, and to assure systematic and psychological organization. This method-whole has been much maligned and is therefore no longer in pedagogical fashion. But for want of an effective substitute it may still be used to advantage in teaching a small unit of logical subject-matter. Much of the odium connected with this form of recitation is due to the misconception that none of these steps can be combined or eliminated if it serves no useful function in a given lesson. The formal steps constitute a suggested plan, not a fixed procedure of teaching.

The organization of the deductive recitation. A deductive recitation is not devoid of plan and purpose. In its complete form the deductive lesson consists of five steps, namely, *the preparation, gathering the data, giving the generalization, making the inference* or *explanation* or *application, the verification.* Whether the recitation is inductive or deductive there must be a preparatory step that calls up necessary old knowledge, establishes an aim, and attempts to stir a motive for the acquisition of the new knowledge or skill. The function of each of these steps can be illustrated by a lesson on the area of a circle.

Preparation: Cite instances in which the area of circular rather than of rectangular surfaces are necessary. Ask for the formula for finding the area of rectangular surfaces. Elicit the inadequacy of $b \times a$ for circular surfaces. Then lead the children to state the object of the new lesson. *Aim* and *motive* are thus established.

Data: All necessary facts must now be gathered. Review meaning of radius, diameter, circumference, and area; have children recall that diameter is twice the radius, and that circumference $= 3\frac{1}{7} \times d$ or πd.

Generalization: The teacher gives the rule, "$3\frac{1}{7} \times$ radius squared," or πr^2.

Application: Examples in the book are solved by applying this law.

Verification: By a construction exercise the answers can be approximated. Compare answers obtained with those in book. Compare area of a circle with area of a circumscribed square.

In teaching long division, the mode of obtaining the least common multiple, or any of the lessons that make the deductive recitation appropriate, these five steps may not all be necessary, nor need they follow the sequence as outlined. It often happens that the data follow the generalization, or that the verification is impossible. Any modification in the organization of the recitation may be made if, in the judgment of the teacher, the effectiveness of the lesson is thereby enhanced. Planless teaching entails extravagant waste of time and energy. Only through planned teaching do pupils grow in knowledge, skill, and power.

Developmental teaching in both inductive and deductive recitations. The strength of both inductive and deductive recitations—but especially the inductive—lies in their dedication to *developmental teaching,* that is, in their attempt to lead children to rediscover socially necessary knowledge and procedures. Both recitations try to tell as little as possible; both seek the development of power through self-expression.

To illustrate: in teaching children to subtract 37 from 62, six dimes, and two cents are shown. It is now necessary to pay a debt of 37 cents, that is, of three dimes and seven cents. It is a simple matter to give my creditor three of the six dimes, but how can I give seven cents when I have only two? The children are now confronted by a problem. What solution can they suggest? A few judicious questions lead them to see the light. Have I enough money? Must ten cents always be in the form of a small silver coin called a dime? How many cents can I receive for a dime? If I change a dime into cents, how many cents will I have? The conversion is made and the children see that now they have five dimes and twelve cents—a sufficient number of each denomination to enable them to pay the debt of three dimes and seven cents. A few more similar problems are solved and the children are led to formulate a procedure of subtracting numbers of two orders where the digit in the subtrahend is larger than the corresponding digit in the minuend.

In teaching division by decimals we may arouse in pupils similar active coöperation. A problem is given in which it is required to divide 24.36 by 6. No difficulty is experienced, for the children know this process. A second problem is now given in which it becomes necessary to divide 24.36 by .6, not by 6, a whole number. Most children begin the division only to stop as they realize that the decimal point in .6 is the cause of their trouble. The teacher elicits from the children that if it were possible to eliminate the decimal point of the divisor they could find the answer very readily. The problem now confronting the class is how to bring about a change in the divisor from .6 to 6. The teacher falls back on facts known by the children and asks the class to convert 1÷2, 2÷3, and 3÷4, into fractions. The board work presents the following appearance:

$$1 \div 2 = \tfrac{1}{2}$$
$$2 \div 3 = \tfrac{2}{3}$$
$$3 \div 4 = \tfrac{3}{4}$$

The class is now asked to convert 24.36÷.6 into fractional form and the answer $\frac{24.36}{.6}$ is obtained. The basic problem, "How to turn .6 into 6," is now recalled and the teacher asks: "What do I do to the denominator by changing $\frac{24.36}{.6}$ into $\frac{24.36}{6.}$?" When the answer, "It is multiplied by 10," is obtained, the children are asked: "If I multiply the denominator by 10, what must I do to the numerator? Have I now changed the value of the fraction?" The work on the board now shows:

$$24.36 \div .6 = \frac{24.36}{.6} \times \frac{10}{10} = \frac{243.6}{6.} = 6.\overline{)243.6}$$

In a similar way children are helped to divide 2.436 by .06. After they are guided to find the quotients in a few similar examples they are led to formulate the rule for dividing by a decimal.

Cautions and suggestions in the use of the developmental method. 1. Inductive and analytic-synthetic in character. The developmental method is essentially inductive rather than deductive and for that reason is analytical in its initial stages; toward the end of the lesson a synthetic process sets in. It is not true that the developmental method is always inductive.

2. Time cost to be considered. Because the developmental method is likely to consume much time, teachers must weigh the reputed gains of developmental teaching against the time cost. Teaching primary facts and skills is essentially a short cut across racial experience. Life is too short for each generation to rediscover what the race knows. Much in denominate numbers can be taught through development, but such a procedure probably involves too extravagant a use of time. No fixed rule can be laid down. In each case the teacher must decide whether the subject is socially or arithmetically significant enough to warrant the time cost.

3. Teaching by development is directed teaching. The developmental method is no undirected search for truth by trial and error. Throughout, the teacher leads the class, by questions and suggestions, to take the correct path and avoid pitfalls in reasoning that yield unwarranted conclusions. Thus,

the developmental method makes of teaching an art of systematically arousing and directing the pupils' self-activity until a socially desired end is attained.

Limitations of the developmental method. The very strength of the developmental method may become its fatal weakness. In guiding and cautioning his pupils the teacher may exercise an intellectual domination over them. Youth is entitled to learn, to a limited extent, through its own errors. While we cannot go as far as Rousseau in advocating negative learning, we must grant that learning through one's own experience has its place in life.

The children of limited intelligence probably profit little by the developmental method. They repeat blindly the series of "therefore's" and then submit to an imposed conclusion. As in authoritarian teaching of an older and severer day the children are once more overwhelmed by empty verbalism.

Values of developmental method. The advocates of this type of teacher guidance urge that the developmental method insures better comprehension—for those able to comprehend —longer retention, and easier recall. A feeling of self-confidence and a mode of thinking are fostered by following a mature line of thought. The pupil thus acquires a degree of insight into number relations that insures power to solve examples varying from the type. Even a capable child who is taught to divide 24.36 by .6 or 2.436 by .06 by rule of thumb is lost when the example is 24.36÷.006. But if the pupil is an active participant in developing the rule he usually has an understanding of mathematical principles which enables him to meet the new situation. Later, in the example 24.36÷600, when he is told to divide, not by 600, but by 6, he can easily be led to perceive that dividing the denominator (divisor) by 100 requires a similar division of the numerator (dividend) by 100. The child sees, easily enough that $600\overline{)24.36}=6.00\overline{).2436}$. The added time required to teach inductively the initial lesson in dividing by a decimal is saved many times over in later situations when the insight into mathematical relationships enables the child to meet situations which are variants from the type.

Lessons for retention and recall. In an earlier chapter the drill lesson is discussed with reasonable fullness. Additional material will be found in Chapter VIII and in the discussion of such topics as diagnostic testing and the teaching of each of the fundamental operations.

PROBLEMS FOR STUDY AND DISCUSSION

See end of Chapter XVIII

CHAPTER XVI

RECITATIONS FOR POWER: STUDY LESSON, PROBLEM, AND PROJECT

In an earlier discussion we took issue with the claims made for the developmental lesson because it imposes too many adult reactions on immature pupils. Children must learn to become independent of their teachers and to rely on their textbooks for new facts and new processes. Unless we teach children how to study arithmetic we fail in a very significant duty.

THE STUDY LESSON

Let us suppose that children are to learn how to find the cost of a quantity of two or more denominations when the cost of the smallest denomination is known, thus: "Find the cost of 6 gallons, 2 quarts, and 1 pint at 6 cents a pint." Instead of assuming full control of the lesson the teacher presents the new problem. The children realize that the situation is new and are reluctant to go ahead. The teacher then brings the matter to a head by asking, "What would you do if you had no one to help you?" The class is led to seek aid in the textbook. A few well chosen questions help the class to plan a procedure:

1. Find the place in our books which explains how to do this.
2. Read the explanation given.
3. Try to do the first example.
4. Check the answer to find out whether we are correct.
5. Do as many other examples as possible, and check the answers.

To be sure, the teacher does not eliminate herself from the lesson, but maintains active supervision of the learning process. This arithmetical process can undoubtedly be taught through a developmental lesson in less time than it takes the children to study it from the printed page. But we are more concerned with developing mastery of study techniques than with the acquisition of new facts or new arithmetical processes.

The textbook evaluated. The "Complete Arithmetic" of a former day has given way to a series of smaller books, each of which covers the work of one, two, or, at most, three grades. In the two-book series we usually find Book I devoted to Grades, 2, 3, and 4, and Book II to Grades 5 and 6. In the three-book series the third book gives the work of Grades 7 and 8. In addition to these two organizations there is a third in which each book gives the work of one school year or even half a school year. The significance of this change from a single book to the series of books lies in the fact that the author has page space to include descriptive and expository material in addition to the usual groups of exercises. Buswell suggests a two-book plan for a given grade: (1) a book given to explanatory material and directions for procedure, and (2) a companion manual of exercises or a workbook.[1] Here we find assurance of sufficient expository material to enable the class to use the text in study lessons. Too many books limit themselves to lists of exercises and assignments. A helpful textbook gives simple explanations of new business situations and new arithmetical processes, diagrammatic representations of quantitative relations, a complete index, and a reasonably full table of contents—all of these provisions being clearly designed to help the child help himself and thus develop mastery of the techniques of study.

A helpful textbook makes for unmistakable economy of time because it is replete with well graded exercises and vital forms of applications to real life. The book should save the teacher the endless hours of formulating the most effective types of practice and genuine problems. Such a book also makes unnecessary the writing of exercises and problems dictated by the teacher. Small books, inexpensively bound and designed as direct workbooks, are probably most desirable under our present system of class teaching.

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[1] G. T. Buswell, "Curriculum Problems in Arithmetic," *Second Yearbook* of the National Council of Teachers of Mathematics (Teachers College, Columbia University, 1927), p. 89.

The textbook should be used freely for verification of inferences drawn during a lesson. In the developmental lessons many conclusions are reached as the lesson progresses, but each conclusion must be tentative depending upon the printed generalizations. The book thus becomes a continuous supplement even in those lessons the teacher is actively directing.

Whether a good textbook in arithmetic should include the correct answers to problems and exercises is a moot question among teachers and supervisors. Clearly not all answers can be given without defeating the very purpose of the questions. An exercise of multiple-choice questions would stimulate no thought in some children if they knew that the answers were accessible. To illustrate:

Tell which decimal has the closest value to the fraction at the left of each line:

⅓	.34	.034	.43	.043
⅝	.855	.06	.087	.6
⅗	.06	.59	.65	1.6

Unless children were under strict supervision a goodly number might seek the answers in the book rather than accept the challenge in the exercises.

Listing correct answers to problems and examples that require planning and careful computations presents a different case. Here the answers enable the child to check his work and to measure his accuracy. When the child is wrong he has a motive for reworking his problem. Frequently a correct solution is evolved by working backwards from a correct answer.

A modern textbook in arithmetic increases its usefulness by the inclusion of many illustrations and diagrams. The pictorial aids make more real those situations that are new or only vaguely known to young people. The diagrams present quantitative relations in a form that quickens recognition. We frequently make glib references to an amazing variety of situations, rarely mindful that these are unknown to the children. A superior class in an urban school was unable to solve a problem concerned with placing fence pickets. To the teacher's surprise none knew what a fence picket was. Too often difficulty in arithmetic arises in the limitations of language or in the scant experience of the pupils. Such difficulties are minimized by an appropriately illustrated textbook.

We have said enough about drill to justify the demand for well planned exercises in our textbooks. The practice material provided on their pages must be well distributed so that the number facts are repeated according to their difficulty. The drills must be functionalized and standardized so that application to real situations is assured, and the class or a pupil can guage progress with reference to carefully established norms. A good textbook also provides exercises that call for oral rather than for written responses.

To these demands of a textbook in arithmetic we must add those characteristics that are expected in a good book on any subject. At all stages the language must be suited to the children; the development must at all points be psychological rather than logical, so that there will be a vital point of contact between the child and the subject. The mechanical requisites of paper, print, format, and binding must measure up to approved standards.

Homework and proficiency in arithmetic. Homework is usually taken for granted. Its reputed values are numerous but still awaiting proof. Teachers give various justifications for assigning homework: it affords added time for practice; it develops habits of study and self-reliance; it reserves classroom time for teaching rather than for drill; it makes for a closer tie between home and school by giving parents an opportunity to learn what the school teaches; in urbanized communities it keeps the children occupied and off the dangerous streets. That homework adds to drill time we cannot deny, but that this after-school-time drill is essential is still to be proved. We do not need more drill periods, we need better distributed individual drills in class under the teacher's supervision. Unsupervised study may develop very undesirable study habits. The child who has found a helpful homework friend acquires a sense of dependence rather than independence. In order to foster genuine coöperation between parents and teachers we must evolve bonds between the home and the school that are more socially significant than homework can be. Although the traffic-filled streets of the city are not safe places for children to play the solution of the problem lies in the establishment of properly supervised play centers —not in the assignment of homework. We have yet to devise a legitimate justification for homework in the elementary school.

The assignment of homework. Properly assigned, homework becomes less of an annoyance to children and therefore less of a problem to teachers.

1. In the elementary grades homework in arithmetic should generally be supplementary to class work. If the class learned to divide by two-place divisors like 19 and 21, or to add fractions with dissimilar denominators, or to compute the area of rectangles, then the homework should require children to perform these operations. Homework thus becomes an extension of the application step of the lesson.

2. We must not make too serious inroads with homework on the child's free time. Most school people are agreed that no homework should be assigned in the first four grades. Ten to fifteen minutes a day of homework in arithmetic in the fifth and sixth years is quite adequate; fifteen to twenty minutes a day in the seventh and eighth years seems a reasonable maximum assignment. Teachers must remember that there are other subjects that seem to have equal claims on the child's after-school time. To reduce its onerousness, homework in the fifth grade should limit itself to practice in fundamental operations and should not include the solution of

problems. It is well to eliminate the homework assignment over weekends and holidays.

3. Every example assigned for homework should be read with the class to discover difficulties originating in the newness of the situation or in the limitations of children's language. These added difficulties should be cleared up in class. It is advisable, therefore, to assign homework immediately after the arithmetic lesson.

4. Homework must be assigned with a view to convenience of correction. Like all other school tasks, homework which is not checked soon degenerates and demoralizes all other written work. Some teachers have the correct homework put on the board and each child marks his neighbor's results. The children who have worked any example incorrectly are helped by those who put their correct work on the board. The books are then collected and are inspected by the teacher, who assigns a grade to the day's work. In this way each child sees his error, learns the correct solution, and receives the teacher's comment—all in little time and with little effort. Any simple system which insures inspection, correction, and individual help may be substituted for the plan suggested.

5. Homework should be kept as a cumulative record in notebooks rather than on loose sheets of paper. At a glance one can thus ascertain a pupil's day-by-day record.

6. Differentiated assignment of homework is highly desirable. Since not all children exhibit the same weaknesses not all need the same practice. In large classes lacking any kind of homogeneity of grouping no such individualization may be possible. In better organized classes children may be grouped and differentiated homework assigned. Those weak in long division are given examples decidedly unlike the ones assigned to pupils who can divide but who have trouble in multiplying with zero in the multiplier; pupils who maintain a uniformly high grade in class work and in class tests may be excused from all homework. Nor is it necessary to require all homework to be written. Factual arithmetic can be learned with-

out writing. A rapid oral quiz will soon show whether the child studied his assignment. Such differentiation of assignment motivates homework and therefore prompts more earnest application to it.

The Dalton Plan of individual progress. There are those who argue that since each must do his own learning it is folly to teach a subject of skills like arithmetic to a whole class. They advocate, therefore, that the whole subject be reduced to a series of carefully graded and progressive units of work that constitute all of the subject deemed worthy of mastery. Each unit of work states its objective, indicates what the child is to read, what process he is to master, and what exercises he is to perform. As a pupil encounters a difficulty he cannot resolve, he appeals to his teacher for aid. Help may be given him directly, or a suggestion may be offered for further reading in the hope that the child may solve his own problem. When the unit of work is completed it is presented for inspection. If it is accepted the pupil goes to the next progressive unit; if rejected, he is told what to rework or what to learn more thoroughly.

Each pupil thus progresses at a rate in keeping with his ability. He does the whole of each lesson and feels, throughout, that each new difficulty is a challenge that he must meet. Each child is assured of abundant practice designed to reduce the techniques of arithmetic to habit. The Dalton Plan may readily be modified: group conferences may be held with those who encounter similar difficulties; occasional developmental lessons may be given on basic processes and thus common errors or common difficulties may be forestalled. This method may be used with the superior pupils only, and the more conventional methods of class teaching reserved for pupils less gifted. Any carefully devised appropriate modification will probably intensify the value of the Dalton Plan for a specific group. We have yet to obtain accurate data, secured through experimental procedure, that will either confirm or deny the values ascribed to Dalton teaching. But those who believe that learning is an individual process in which the teacher can exercise at best only a benevolent superintendence, may feel justified in following the Dalton Plan very seriously, but, at all times, very critically.

THE PROBLEM METHOD

We must distinguish the problem method of teaching from solving arithmetical problems. The latter refers to a narrow range of techniques required to ascertain quantitative relationships; the former, to a teaching procedure that follows the natural processes of the mind when it is engaged in meeting a difficulty. In teaching, the term *problem* identifies any challenging situation, any difficulty that invites solution, hence the problem method is employed as frequently in teaching history, language, or geography as in arithmetic.

Organization of the problem recitation. Thinking is generally initiated by a need for readjustment in old experience. Undisturbed routine tends to discourage thought. But when a new factor arises in the accustomed regimen of life, thinking is aroused. The child who can multiply by a number of three orders containing any three digits except the cipher does a minimum of thinking when the multiplier is 246 or 375 or 892; but the moment a zero is introduced in the multiplier (thus, 802) a new condition arises and the mind's power to reason is awakened. Steps, more or less definite in function, are recognizable in an act of reasoning. These are (1) a conflict in experience; (2) recognition of a problem; (3) attempts at solution; (4) evolution of the final explanation or procedure; (5) verification of the conclusion. The scope and function of each of these steps are revealed in the following discussion of a lesson.

In teaching multiplication by a decimal certain problems were presented, and the children were asked to indicate the mode of solution. They suggested that $24.54 be multiplied by 8 to obtain the answer for the first, and, for the second,

$72.36 be multiplied by 12. Since they knew how to multiply dollars and cents by a whole number they obtained the answers with no trouble. The numbers 8 and 12 were then changed to .8 and .12 respectively, and the children were again asked to indicate the mode of solution. It was clear to them that $24.54 × .8 and $72.36 × .12 would give the required answers. When asked to perform the operations, some children stopped at once; others continued beyond the initial point, but all failed to complete the process. The stages of incompleteness are shown below:

$24.54	$24.54	$72.36	$72.36
.8	.8	.12	.12
	$19 632		$86 832

A few questions elicited from the children that these multiplications involved a new condition and that the objective of the day's arithmetic period must be, "To find out how to multiply by a decimal." We have here (1) a conflict in experience, and (2) the recognition of a definite problem.

"What are probable ways of obtaining the answer?" the teacher asked. All agreed that $24.54 must be multiplied by 8 and $72.36 by 12, but that the products thus obtained would not be the correct answers. The children were then led to conclude that if they knew where to place the decimal points in the numbers 19 632 and 86 832 the entire difficulty would be removed. Each child was then asked to find the place for the decimal point. Some pupils thought it belonged between the 9 and the 6, thus, $19.632; and between the 8 and 6, thus, $8.6820, "because if you add the places in the two numbers you get three places in the first example and four in the second." Others objected to adding the places "because these are examples in multiplication, not addition." These children suggested the answers $196.32 and $8.6820, because "in the first example we have two places and one place, 2 × 1=2, but in the second, we have two places and two places, 2 × 2=4." The lesson had, thus far, progressed through the next two steps, namely, (3) attempts at solution, and (4) evolving the final procedure.

The children now realized the need of testing their respective conclusions. Since they knew no method of verification the teacher suggested the solution by common fractions. This was attempted and the following results were obtained:

$$\$24.54 \times \frac{\overset{4}{\cancel{8}}}{\underset{5}{\cancel{10}}} = \frac{\$98.16}{5} = \$19.632$$

$$\$72.36 \times \frac{12}{100} = \frac{\$868.32}{100} = \$8.6832$$

It was evident that those who "added the places in the two numbers" were correct. After two more examples were solved and the answers tested, the rule for pointing off in decimal multiplication was established.

In lessons which will be outlined in later discussions and in the study of problem solving we shall have many more illustrations of problem recitations.

The caution that was urged in the use of other recitation patterns must be repeated here. There is nothing unalterable in the sequence or in the number of these steps. Teachers must rise above the method if it is to serve classroom needs most effectively. Each of these recitations is no artificial organization that has been evolved by teachers of methodology, but it is rather an attempt to incorporate in formal instruction the psychological processes that function in normal and natural life.

Project Teaching

In our daily class teaching we usually present preparatory experiences, facts and skills that, in our judgment, will be serviceable in adult life. In vital learning the child participates in life rather than prepares for it. Project teaching advocates the attack upon units of purposeful experiences taken from life, rather than systematic mastery of graded units of subject-matter.

Instead of random writing and reading of numbers, project teaching would begin with an actual need in present-day living. Young children would be invited to make up a directory of their friends by listing names, addresses, and telephone numbers. Here we have a natural task in which are correlated spelling, penmanship, reading and writing of numbers.

Fifth-grade pupils are expected to continue practice in adding and subtracting whole numbers. A series of isolated exercises may have preparatory value in building up skills to a necessary degree, but performing the arithmetical operations of a day's business of the school bank attains the same aim in a more vital way. The deposits and withdrawals afford plenty of practice in addition. Similarly, tracing the investment and the receipts of the man going into a small business, computing the cost and the assessments of a local improvement, keeping a personal account book, or performing the bookkeeping involved in checking the family budget afford opportunity for teaching necessary arithmetical techniques.

But the very nature of the subject of arithmetic makes it an unpromising field for project teaching. In the last analysis the suggested projects are not really genuine needs that arise in the child's life; they are imposed situations because we consider them likely to arise in adult life. The child is not going into business and he probably cares little about the cost of local improvements or the modes of assessments to raise the needed sums. The family budget is no vital concern of the child of ten, and his personal account involves items too few and too small to give real practice in arithmetic. Are these purposeful situations more real to young children than troy weight or the apothecaries' table so common in the books of an older day? The child perceives the usefulness of the tasks listed above, and they are thus motivated; but all motivated teaching is not project teaching.

The highly logical subject-matter of arithmetic, with its hierarchy of skills, makes a graded development absolutely essential. The child learning to write numbers may still be in the stage of two-place numbers. To make an address book may require the reading of numbers of three or four places. Also, the project may afford interesting and useful reviews, but it does not insure the organization of graded material, which is essential in facilitating the learning of arithmetic.

Project teaching is rich in opportunities for very vital activity, but we must recognize its limitations in a subject that lacks the social content of history, geography, or literature. Few useful projects in arithmetic live up to the promise in the definition: a unit of purposeful experience. The experience is purposeful to the teacher but only infrequently to the child.

PROBLEMS FOR STUDY AND DISCUSSION

See end of Chapter XVIII

CHAPTER XVII

ORAL AND WRITTEN ARITHMETIC

As a rule the recitation in arithmetic consists of two parts, oral and written. These differ in purpose, in spirit, and in method.

Oral arithmetic. *Length of the oral period.* In general, a day's work in arithmetic should include the type of work usually referred to as oral arithmetic. To it we should devote a minimum of one-third of the entire period. On days when a new process is taught we may give considerably more time to oral work; in fact, the written arithmetic may be entirely eliminated.

Aims of oral arithmetic. The old distinction between oral and written arithmetic, *per se,* is difficult to maintain. The terms are somewhat ambiguous, for who can tell how oral is oral arithmetic? What if a set of well selected examples, presented on a printed page or a chart, are to be worked out without paper and pencil? Does this constitute oral arithmetic? What if the answers are written instead of spoken? Is such an exercise oral or written arithmetic? In strictly oral work only one child gives an answer. Unless the others write their answers, the check-up is of necessity limited. It is wise, therefore, to permit the two forms, oral and written—especially when modern practice or workbooks are used—to mingle freely. The chief concern of the teacher should be the attainment of the characteristic objectives of oral arithmetic.

The following considerations should determine the relative amount of oral and written work in a given lesson:

1. Give rapid drill designed to habituate a fundamental process, a mode of thinking, or a set of facts. In teaching how

to find the L.C.D. by inspection, how to find the cost of quantity x when the cost of quantity y is known, or to recognize instantly the per cent equivalent of certain common fractions, the work is of necessity oral.

2. Increase the amount of work to be done in a given period. Let us assume that children have just learned how to compute the cost of x articles at y cents a dozen. To insist that the children's applications be made in written form is to limit the amount of their practice. It is wise, therefore, to present to the class ten or fifteen problems on a large chart or on a mimeographed sheet, and then to call for the oral solution of each. In the time required to perform three examples in written form ten to fifteen can be solved orally.

3. On the other hand, when we deem it necessary to inspect much work done by each child or to give children practice in independent work, the written form will of necessity be used. On occasions when the class exhibits a readiness to work together in good spirit the lesson becomes oral rather than written. Invariably the nature of the task determines the degree of "oralness" or "writtenness" of the lesson. "Devotion to oralness or mentalness as such is simply fanatical."[1]

The organization of oral arithmetic. The following plan used in a sixth grade indicates the scope of the oral work:

The Fundamental Operations. Daily 10 minutes.

- *a.* Daily—Rapid column addition, adding statistics found in appendix in geography. Rapid subtraction of type, 27 from 82. Rapid multiplication and division by usual circular devices. Time, 5 minutes. And,
- *b.* Monday—Addition and subtraction of common fractions.
  Tuesday—Multiplication and division of common fractions.
  Wednesday—Four operations in decimals.
  Thursday—Finding 1%, 2%, 2½%, 3%, 3⅓%, etc., of a number. Time, 5 minutes.

Each day ten minutes were devoted to the mechanics of arithmetic, five minutes every day to whole numbers, and five minutes to a more specialized drill in the mechanics of the fractions and percentage. Examples like $.2\times.1$, $.2\times.01$, $2\div.2$, $.2\div.02$, $.02\div.2$, $3\frac{1}{2}\%$ of \$250, 1% of \$500, 1% of \$50, 1% of \$5, $\frac{1}{2}-\frac{1}{4}$, $\frac{3}{4}-\frac{3}{8}$, $\frac{1}{2}\times\frac{3}{8}$, $\frac{3}{8}\times\frac{4}{5}$, etc., were solved orally with a degree of accuracy and speed that hardly seemed possible at the beginning of the term.

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[1] E. L. Thorndike, *Psychology of Arithmetic* (Macmillan Co., 1922), p. 263.

The conduct of the oral arithmetic. Spirit is infused into oral work by the alertness which the teacher displays in giving problems and exercises to the class. The materials for oral solution must not be read to a class from a book. The teacher must prepare her work, select her numbers, and then give the exercises to the children with seeming spontaneity. To save time children may be asked to read silently examples from the printed page or from a mimeographed sheet. But always time must be limited, and children must be held to strict accountability for responses. To state problems or indicate exercises and then call on volunteers for the answers discourages attention and concentration. Every child must feel responsible for all answers within the time set by the teacher. It is the oral period which sets the pace for the written work. Many teachers look upon the oral arithmetic as the mental setting-up exercises. The correlation of proficiency in oral and in written arithmetic has been given by one investigator as +.60.[2] We cannot, of course, interpret this fact too statistically because of the difficulty of making an accurate classification of arithmetical work into exclusively oral and exclusively written. It is well to note that the correlation is significant.

The period of written arithmetic. That part of the lesson known as *written arithmetic* is merely an extension of the application step of the oral work. Effort is directed to solving problems of the type that was taught; the numbers are now so large that they make written computation necessary.

Aims of written arithmetic. Throughout the written work three aims must ever be kept in mind: accuracy of computation; speed consistent with accuracy; proper algorism—logical and sequential arrangement of the steps in the solution and neatness of work.

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[2] E. E. Wilson, "Correlation Between the Oral and Written Work of Pupils in the Fundamentals of Addition," *School and Society*, Vol. 5, March 10, 1917, p. 300.

Written arithmetic to be kept as a collective record. It is suggested that written work should be kept in a special notebook or in a large envelope in which the daily sheets are arranged in chronological sequence. A collective record shows at once the progress or the retrogression of any child in accuracy, neatness, and effort. The child whose work is improving experiences a sense of pride in his accomplishment, while his less earnest neighbor is constantly confronted by evidence of decreasing merit. The supervisor who inspects the cumulative records of three of the ablest pupils, of three average pupils, and of three slow pupils can tell in a general way the character of the curve of progress of the entire class.

PROBLEMS FOR STUDY AND DISCUSSION

See end of Chapter XVIII

CHAPTER XVIII

ELIMINATING WASTE FROM THE RECITATION

It is not uncommon to find the recitation in arithmetic opened by the assignment of one problem. The children attack the solution with avidity. After three minutes of work children begin putting down their pencils and indicate by exaggerated posture that they have the answer. They sit in idleness waiting for their slower classmates. After several minutes elapse all pencils are ordered down and a designated pupil proceeds to the blackboard to make a copy of his work and explain his solution.

Questions are asked of backward children and weak places are bolstered up. At least five or six minutes are required for this explanation. Section monitors are now given the signal to mark their classmates. This done, the first example is completed at a total time cost of about fifteen minutes. A second problem is treated in much the same way and the half-hour devoted to written work is at an end. Here is waste. In such classes the author found that often about 40 per cent of the children solved both problems correctly in a maximum of six to seven and one-half minutes. These pupils lost over twenty minutes waiting for their slower classmates and were subjected to the deadening experience of listening to explanations of processes well understood by them. We must now turn to the problem of economy in the recitation, the causes of waste, and the appropriate remedial suggestions.

Causes of Waste in the Recitation	*Remedial Measures*
1. Time is often lost by incompetent class management. A con-	1. A monitorial system must be devised that will permit the distribution of necessary material before the class arrives.

Causes of Waste in the Recitation	*Remedial Measures*
siderable portion of the recitation is consumed in distributing material and in ruling papers for uniform arrangement. Teachers often force children to sit idly by as they draw elaborate diagrams or write lists of figures on the blackboard. Such losses of time are inexcusable.	Children should be encouraged to rule papers before the lesson begins. The rulings must be exceedingly simple. Before the class assembles necessary diagrams and drill material must be prepared on large cardboards or on slate cloth and hung, ready for class use.
2. The dictation of problems with unnecessary phrases is another practice that makes for waste in the recitation.	2. *a.* The obvious remedy—to omit all unnecessary phrases—is not sound. Children must be taught to listen intelligently and discriminatingly and then put down the essential elements of a problem. *b.* Examples should be worked from books as far as possible. *c.* Mimeographed sheets containing suitable examples may be prepared and distributed among the children. *d.* Examples cut out of old books may be pasted on cards. These are distributed and the children sent to the blackboard.
3. Time is often lavishly spent working out examples with long numbers and difficult combinations.	3. *a.* If the purpose is to give drill in the fundamental operations, a mechanical or abstract exercise should be given, for example, 3426 × 786; but when problems are assigned, the aim is to develop thought in arithmetical situations. The numbers chosen must be such as to avoid arduous division and multiplication. This suggestion is not intended to produce an ultra-simplification of arithmetic, but rather to conserve a maximum of time for the specific purpose of a given type of lesson. *b.* Visualize the process. When an important and difficult type is taught,

Causes of Waste in the Recitation

Remedial Measures

effective drill in the mode of solution can be given by omitting the mechanical operations altogether and merely visualizing the solution. Let us assume that the new type is in unitary analysis: to find the cost of x units, knowing the cost of y units, the numbers all in fractional forms. A mimeographed sheet containing about twenty-five of these problems, all well graded, is distributed. The exercise reads as follows:

If 3½ yards cost $10.50, what will 5¼ yards cost?

What must I pay for 18⅛ yards at the rate of 5½ yards for $7¾?

Paid, $17.25 for 8¼ rolls of cloth. Find the cost of 12⅓ rolls. Etc.

The children are told to read the first problem; the recitation then follows: "Pupil A, what is sought? B, what is given? C, what is the first step? D, what is the second step? E, how do you perform the first step? F, approximate the answer. G, how do you perform the second step? H, approximate the answer. J, how would you test the accuracy of your final answer?" Each problem is subjected to oral analysis, to visualization of its process, to an approximation of its results, and to a mode of checking, but not a single step is worked out on paper or on the board. In twenty minutes about twenty examples are performed and the class is given vigorous drill in the rationale of this type of problem. A few of these problems may be assigned for homework and as many others as may be necessary are solved the next day in class. This process of visualization is not recommended as a daily procedure.

Causes of Waste in the Recitation	*Remedial Measures*
4. Time is wasted for some pupils by work that is not apportioned according to ability. To assign a set number of examples for all children involves waste for those of distinct ability.	4. Children must be made to feel that they are expected to work at maximum capacity. It is advantageous, therefore, to group children into groups A, B, and C, according to their capabilities in arithmetic. Groups A, B, C, are given examples 1 and 2; group B is given an additional example; group A is given two additional examples. In other words, group A is expected to do examples, 1, 2, 3, and 4, while group B does 1, 2, and 3; group C is not expected to do more than the two examples. Examples 1 and 2 are type forms, example 3 is a slight variation, while example 4 taxes the child's ingenuity for its solution. Work is thus apportioned in quantity and quality, according to ability and all children are assured of practice in the type form. Examples 1 and 2 are explained in detail. While children in Group C are correcting their errors, those in A and B who have incorrect answers to examples 3 and 4 attend to the explanations. A modification of the plan is to send all children in Group C to the blackboard while those in B and in A are in their seats. Work is assigned as above or the problems given the C children at the board may be entirely different from those assigned to the A and B pupils. The teacher gives the first part of the period exclusively to the children at the board. While the C children correct their errors the teacher takes up the work with the A and B groups. Any variation which seeks to assign work according to capacity and break the lockstep system in instruction is superior to the old system of uniform assignment and uniform requirement of all children.

Causes of Waste in the Recitation	*Remedial Measures*
5. A recitation without purpose or aim is responsible for dissipation of energy and time.	5. In planning a recitation there must be a very specific aim governing the entire period and determining the nature and the quality of work assigned. To fill the period with problems or exercises is not enough. The purpose may be to increase speed, to secure greater skill, accuracy in multiplying by three-place multipliers having a middle zero, or to develop mastery of a given type of solution. Every step in the recitation and every exercise assigned to the class must be designed to achieve the specific aim. Any divergence from this aim results in direct loss of energy and time.
6. Undifferentiated and non-individualized drill entails as much waste and produces as much inefficiency as insufficient drill.	6. In earlier discussions on drill and habit formation the dangers of undifferentiated drill were emphasized. Early in the learning processes we must discover through diagnostic testing the specific weaknesses of each pupil. The drill provided for each child must be in keeping with the weaknesses revealed in the diagnostic test. What we need is not more drill, but more differentiated drill so that practice and need are in close correlation.
7. Indefiniteness of assignment is a serious cause of time waste.	7. Assignments in arithmetic, whether for class work or homework, must be analyzed carefully by the teacher. We must be certain that unnecessary difficulties are not allowed to creep in, that the language and the conditions set forth in the problem are not beyond the pale of the child's experience, and that the problems are applications of principles taught in the class. Improperly assigned work robs children of valuable time and leads to unwarranted discouragement or resentment born of unreasonable demands.

Causes of Waste in the Recitation	*Remedial Measures*
8. Economy requires that the attention of all should not be directed to explanations needed by a minority of the class and that formal analysis of a type problem, well understood, should not be continued.	8. After a problem has been completed those children who made mistakes should be called to the board in a group and the example in question explained to them while the rest continue with the next problem. The second example should be treated in the same way; only those who failed in the solution should be given the explanation while the others are permitted to invest their time to greater profit in additional work in arithmetic or in some form of study work in other subjects.
9. Failure to use shortcuts entails unnecessary expenditures of time.	9. In framing problems the teacher should select those numbers for multiplication or division which invite the use of short cuts. Instead of asking for the area to be paved if a gutter measures 54 × 129 feet, the numbers should be changed to 54 × 125 so that the child will be encouraged to add three ciphers and divide by 8. If short cuts are to be of significant value, their use must become a matter of habit. The common practice in which children perform their manipulations by the long process and use the short cut only when the teacher says, "Be sure to do this by the short way or you will be marked wrong," is an utter waste. Short cuts must be so taught that children will use them almost mechanically. Then and only then are they sources of economy of time and labor.
10. Another cause of waste in a recitation is the assignment of problems that are outside the scope of practical business usage.	10. It was emphasized in our earlier discussion that problems must be real; that the conditions they set forth must make solution seem necessary; that they must constantly reflect the practices of commerce and industry. To violate this cardinal rule is to reduce the recitation in arithmetic to puzzle solution and to divorce school work from life.

PROBLEMS FOR STUDY AND DISCUSSION

1. List the incidental number facts which first-grade children may learn from dramatization of a suitable story, and indicate the essentials of teaching procedure. Illustration: "The Three Bears," first, second, third, three largest, smallest, etc.

2. Select three games suitable for children in the first two grades. List the number facts children may learn from each of these games. (Select real games, not specially designed number games.) Characterize the method of directing the learning of these facts.

3. List all the number experiences that arise naturally, in the course of a day in school. Illustration: Time to leave for school, time when school begins, the number on the class door, and so forth. What use can the teacher make of a complete list of this kind?

4. Select one of the following projects and indicate, step by step, its probable progress in your class:
   (*a*) Making a calendar at Christmas time.
   (*b*) Making jelly.
   (*c*) Making a doll house.
   (*d*) Planning, making, and caring for a garden.
   (*e*). Making a directory of the members of the class.

   Give for the project selected a reasonably complete list of probable arithmetical and social outcomes.

5. A primary teacher numbered each seat and gave each child a card with his seat number on it. Each day, as the children entered, they were met by four duly appointed ushers who insisted on seeing the card before showing the holders to their seats. (*a*) What was the teacher trying to teach? (*b*) Describe a more formal procedure for attaining this aim. (*c*) Which of the two procedures do you prefer? Why?

6. Examine three courses of study in arithmetic. What type of lesson patterns, if any, does each course encourage the teacher to follow? What is the basis of your inference? Make a similar analysis of two textbooks in arithmetic, one designed for the early grades and another for Grades 5–6.

7. "Thoughtful teachers will admit that their instruction is more effective if the basic concepts and the fundamental principles are taught by means of a series of projects or activities that challenge the best effort and thought of pupils through the control or drive of vital purposes. Several difficulties limit the use of the project method" (from *Contemporary Guide in the Teaching of Junior High School Mathematics*, p. 95).
   (*a*) Plan an argument designed to change the convictions of a teacher who denies the thesis contained in the first sentence.

(*b*) What are the difficulties referred to in the second sentence? Give an illustration of each.

8. Using the content of Chapters XV–XVIII, prepare a test consisting of two parts: Part I, new-type test of multiple-choice, completion, true-false, and matching questions; Part II, six questions designed to arouse critical reactions and requiring short essay answers. Justify the organization of your test and each of the questions in Part II.

PART VI

SPECIALIZED TEACHING TECHNIQUES IN ARITHMETIC

CHAPTER XIX

TEACHING THE NUMBER CONCEPT: COUNTING AND MEASURING

What number facts do children bring to school? The old question, which should be taught first, counting or measuring, is answered by the children. They come to school able to count and possessed of more number facts than many have suspected. A study made of 1290 children entering school reveals the following facts:

> Ninety per cent of these children could count up to 10; 60 per cent of them could count up to 20; 50 per cent of them, up to 40 by 10's; 50 per cent of them could give any number of objects from 1 to 10; over 25 per cent of them could combine simple numbers like 3 and 2.[1]

These children had already begun their number work by counting and some had taken the next step. To children of this age counting is, of course, a mere rote process: names follow in a fixed order. They do not understand the meaning of any of the numbers whose names they call off glibly. Without instruction they continue to add to their stock of number facts and come to know $3+1$, $2+2$, $5=2+2+1$, $5=4+1$, $5=3+2$, and $3-2=1$. They do not recognize these as formally as we have set them forth, but they do know them when need arises. Left to themselves, children will learn by themselves, but the fortuitous process is slow, unorganized, and very incomplete.

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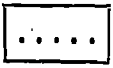
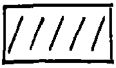
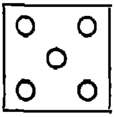
[1] Buckingham and MacLatchey, "The Number Abilities of Children When They Enter Grade One," in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education, Part II (Public School Publishing Co., 1930), Ch. iv.

The Meaning of Number

The concept of number is foundational in the teaching of arithmetic. There are many interpretations of number and each of these influences methods of teaching the basic number facts.

Number as symbol. The earliest conception of number which influenced teaching holds that number is merely a symbol for a group. Arithmetic, therefore, was a purely utilitarian subject which gave all the number symbols and the rules governing their manipulation. The *figure* 5 was made synonymous with the *number five* and was taught by a severely deductive process based on imitation and endless repetition. The arithmetician was an artisan and the teachers of arithmetic were members of the *Rechnenmeisters* guild. Since number was regarded as a symbol, no educational influence was sought from the teaching of arithmetic. Although this explanation of number is old, it is not necessarily obsolete, for much of the rote teaching, with its emphasis on imitation and repetition, gives evidence that number and symbol are still identical in the minds of some teachers.

Number the result of sense experience. Pestalozzi was the first educator who sought to establish a psychological basis for number. To him, number was more than a symbol; it was a mental image derived from sense experience. Four books are seen, four coins are touched, four sounds are heard—these sense impressions are recorded by a one-to-one correspondence, and we become aware of *four*. Just as the facts, *red, rough, warm, round* are acquired through the senses, so, too, it is asserted, number arises from a repetition of sense experience.

Conclusions for teaching. The pedagogical maxims taught by Pestalozzi make clear his conception of number as a sense fact. His first maxim for teachers is "*Number before figure.*" He differentiated clearly between $\boxed{\cdot\ \cdot\ \cdot\ \cdot\ \cdot}$ or $\boxed{/////}$ and the symbol 5. The mental picture which images the group of dots or lines is the number; the written symbol, 5, is the figure. To teach the figure before the picture  or  or  is clearly imaged in the mind, is to teach a sign before the object which it represents.

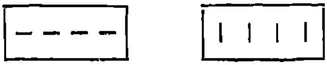
A second pedagogical maxim made famous by Pestalozzi is, *"All number must be taught orally and objectively."* If number is a sense fact, children must learn number by seeing, hearing, and touching every variety of illustrative aid—papers, beads, strings, chairs, pupils, pencil taps, bell sounds, hand claps, etc. To insure this perceptual basis for number and guard against too early an introduction of figures he emphasizes, for the first time, oral teaching of number.

Pestalozzi observed that counting, for some reason, appealed to children as much as manipulating concrete materials. He therefore decides that *all number must be taught through counting* all possible objects and sounds in the environment. In seeking to teach number exclusively through counting, Pestalozzi's practice naturally suffers from the weakness of any extreme position. To Pestalozzi must be given credit for having introduced the perceptual basis in number teaching and for having indicated the disciplinary value of arithmetic.

The origin of number in rhythm. A conception of number closely related to that evolved by Pestalozzi is found in G. Stanley Hall's belief that "counting is the rhythmical punctuation of the stream of consciousness." The human mind is extremely sensitive to rhythmic repetitions and is agreeably affected by them. Where rhythm is absent in repeated experiences we introduce it: witness the regular nodding of head or tapping of foot in counting; we hear the ticking of the clock as *tick* tock, *tick* tock, or tick *tock,* tick *tock;* a more pronounced movement of the rocker is introduced at each second or third rocking of the rocking-chair. Repetitions of certain numbers are looked upon with favor while others are not attractive. Shopkeepers find that they can sell more articles when the price is set at 47 or 49 cents than at 45 cents, more at 19 cents than at 16 cents. The "5- and 10-cent" stores or the "3-, 9-, and 19-cent" stores have an asset in their very names which include numbers that command attention.

How to teach number through rhythm. If we accept the view that number has its origin in rhythm and that "counting is the rhythmical punctuation of the stream of consciousness," the rhythmical series must be the starting point of the lesson. The series, 1, 2, 3, 4, 5, . . . 2, 4, 6, 8, 10 . . . 3, 6, 9, 12, 15 . . . must be seen, heard, performed, and felt. After sufficient repetition of these rhythmic series, the number names may be given. These names should be associated with a series of objects that can be seen and touched, for example, with a series of movements like hopping, stepping, or hand clapping, and with a series of sounds until the number names take on full meaning and become habituated. Throughout this mode of teaching, the pupil is assured of objective presentation or of a perceptual basis for his knowledge of number, and of much oral practice. The danger is obvious: children may learn the number names but make no association with actual experiences nor perceive any of the basic relationships. They may repeat, "one, two, three, four, five, six," or "two, four, six, eight," or "three, six, nine, twelve," but may be unable to recognize six objects, or to understand that two and two, or two two's, make four. Rhythmic experiences should be given in teaching number facts, but not to the exclusion of other procedures.

Number as ratio. Pestalozzi and his disciples, lost in the psychology of perception, regarded number as a sense fact. They taught that when a number of things is perceived, the mind apprehends number. A critical analysis of the number idea reveals that the mere presence of a multiplicity of objects does not give rise to number concepts. For example, the savage may catch many fish, but his knowledge of number seems to have no relation to the size of his catch. Children who see five or nine things do not necessarily know five or nine. Give them six objects and ask them to count. They say, "one, two, three, four, five, six, seven, eight," keeping right on beyond six. Clearly there is no identification of the word, *six*, with the quantity it represents. Their counting is a *mechanical* or *rote* process, not a rational reaction. Show children of limited schooling the following arrangement of equally sized lines and ask them which is more; they usually select the first.



When children are asked to show five fingers they usually count as they point to a finger at a time—"one, two, three, four, five," and then point to the last finger as five. They mean, of course, *fifth*, but they say *five*. Five never means five one's, two and three, three and two, four and one, two and two and one. The first conception of number is usually *ordinal*, not *cardinal*, referring to place rather than to quantity.

Number is not learned by mere sense contact with a group, but by analyzing the group and noting the relationships among the constituents of the original group. *Only as we go through the process of comparison, of analyzing and synthesizing, that is, of measuring, does the complex idea of number arise.* Number is the total of all the relationships in a group. Number is an idea which the mind evolves and applies to experiences; it is not an idea that corresponds to mere multiplicity of experiences in the world outside. Number is a rich interpretation of sense experience, not a mental image of sense experience.

Number, therefore, is an expression of relationships that exist among quantities. It summarizes the ratios that exist between measuring units and the whole quantity. Thus, ten refers to 10×1 or 2×5, especially when we think of money. Six may refer to 3×2 or 2×3 or 6×1, depending on circumstances. Nine may refer to 9×1 or 3×3, depending on the unit of measure. When we have three yards, we may think 3×3 (3×3 feet) or 9×1 (that is, 9×1 foot). Thus, we repeat, a child who says, "one, two, three, four, five, *six*," does not necessarily know the number *six*. Not until he has seen the relationships, 3×2, 2×3, 6×1, ½ of 12, 5+1, 1+5, 4+2, and 2+4 does he really come to a reasonably full understanding of the number six. Mere mechanical counting or deriving each new number by adding one to the preceding number does not give an understanding of number.

In developing a number concept *units* of measure, not a *unit* of measure, should be used. Since a unit of measure is *any* quantity used to compare quantities of the same kind, *one* or *unity* cannot and must not be the only unit of measure. In actual number experiences one is usually not the unit of measure. Assuming that the morning newspaper costs two cents and the evening paper, three cents, then we think of 6 cents as expressing the value of three morning papers or two evening papers. Carpenters use a two-foot rule or a ten-foot rule, rarely a single-foot rule. In purchasing many articles we think in terms of a dozen; we buy a dozen eggs or a half-dozen oranges. In dealing with population figures the unit is 100 000 or 10 000, but rarely a single person. In making payments of $.75 or $1.25, the unit is probably 25 cents, not 1 cent.

Applications of ratio concept of number. The ratio conception is extremely useful. It does more than give a richer number concept. Every short cut is an application of the ratios in number. In the example, multiply 8524 by 25, we teach the child to multiply 8524 by 100 and then to divide the product by 4. Are we not utilizing the relationship or the ratio, 100:25=4:1? Unless this relationship is explained the short cut is taught mechanically, as an arbitrary fact. We shall have many more illustrations in our later discussion of short cuts.

In the analysis of problems the recognition of the ratios involved often saves much time and labor. Consider the following examples:

1. At the rate of 6 for 9 cents, what will a dozen cost?
2. Bananas sell 3 for 5 cents. How many can I buy for 25 cents?

Older solutions sought the cost of one and found the answer by unitary analysis. The relationship between 6 and 12 in the first example and between 5 and 25 in the second was overlooked. These examples which are difficult to solve by unitary analysis become almost obvious by ratio analysis. The contrasting columns show this clearly.

Example	*Unitary Analysis*	*Ratio Analysis*
1.	6 cost 9 cents 1 costs ⅙ of 9 cents or 9/6 or 1 3/6 or 1½ cents. 12 cost 1½ cents × 12 or 18 cents.	6 cost 9 cents 12 = 2 × 6 ∴ 12 will cost 2 times as much as 6 Ans. = 9 cents × 2 or 18 cents
2.	3 bananas cost 5 cents 1 banana costs ⅓ of 5 cents or 5/3 cents. For 25 cents one can buy as many bananas as 25 cents will contain 5/3 cents, hence 25 ÷ 5/3 =	3 bananas cost 5 cents 25 cents is 5 × 5 cents ∴ for 25 cents we can buy 3 bananas × 5 or 15 bananas.

$\overset{5}{\cancel{25}} \times 3/\cancel{5} = 15$, the number of bananas.

To solve these simple examples by unitary analysis the child must have considerable knowledge of fractions and a mastery of such operations as division of fraction and cancellation. By the ratio analysis mastery of the number relationships found in the multiplication tables is quite adequate.

In approximating answers the ratio analysis is most helpful. Let us assume that the cost of 30 inches or 5/6 of a yard is known to be $4.00. The question in the mind of the purchaser is, "Can I afford this material of which I now need 6 yards?" Comparison of two solutions reveals the superiority of ratio analysis in all approximations.

Unitary analysis:

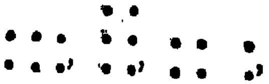
5/6 of a yard costs $4.00.
⅙ of a yard costs ⅕ of $4.00 or $.80.
6/6 of a yard or 1 yard costs $.80 × 6 or $4.80.
6 yards cost $4.80 × 6 or $28.80.

Ratio analysis:

I need 6 yards or $\frac{36}{6}$ yards.
I know the cost of $\frac{5}{6}$ of a yard.
$\frac{36}{6}$ will cost a little more than 7 times the cost of $\frac{5}{6}$ of a yard.
7 × \$4.00 = \$28.00. Hence 6 yards will cost a little more than \$28.00, a fair approximation.

What a difficult problem this becomes by the unitary analysis—how simple by the ratio analysis!

All these applications of the ratio concept of number show clearly that only as we relate things in a series do we approach the number concept. Six as • • • • • • is an indefinite and very vague quantity; it represents plurality and nothing more. Six as



expresses a variety of relations and becomes number in its vital sense.

Ratio idea results from measurement rather than from counting. Older teaching practices, we saw, sought to build up an understanding of number by counting, that is, by applying a set of names in series to a succession of single objects. In current methods of teaching number, measuring is stressed. The child is led to discover the relationship between the whole quantity twelve and specific units of measure, *two, three, four,* and *six.* An analysis by contrast of the two processes, counting and measuring, may sharpen the meaning of each.

Counting	*Measuring*
1. A mere mechanical application of names to objects, one at a time, hence at best, an attempt to discover *how many* times a unit has been repeated.	1. An attempt to discover the relation between a quantity and a measuring unit. The aim is to ascertain *how much* of a quantity.

Counting	*Measuring*
2. A synthetic process as each number is built up by adding one.	2. An analytic process as each number is evolved by discovering the relationship bebetween the whole quantity and the measuring unit.
3. A rhythmic sense in time and space is developed.	3. A sense of equivalence is developed: 3 two's, 2 three's, and 6 one's are all equal to six.

An illustration will cap this distinction. Let us require the child to discover how many nickels in 20 cents, how many yards in 12 feet or how many quarts in 4 pints. Each quantity is analyzed in terms of a unit of measure, that is, it is measured by 5, by 3 and by 2 and the results obtained are 4, 3, and 2 respectively. In each case the process was analytic, and through equivalents built up an understanding of the whole quantity. But in noting how often each unit of 5 cents or 3 feet or 2 pints is repeated, the child is counting and discovering *how many* by a process of rhythmic repetitions.

Despite this distinction McClellan and Dewey insist, "All counting is measuring and all measuring is counting." Taken literally, this statement is not true, especially when applied to mechanical counting, for the two processes are not the same; regarded as a summation of final results, it indicates the mutually supplementary relationship existing between counting and measuring.

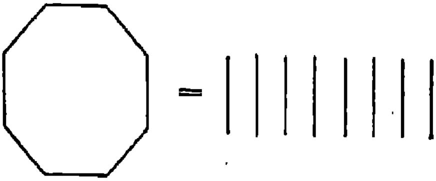
The three kinds of units of measure. The types of measuring units must be recognized. The first is the *undefined unit* which measures by "more or less," by "much or little," by "many or few." The second is the *defined unit,* the pace, the hand, the step—measures that vary within certain limits but which convey a reasonably satisfactory idea of dimension. The third is the *standardized unit,* the recognized unalterable unit like foot, yard, meter, gallon. In the earliest grades the idea of ratio, of magnitudes, must be built up by using the undefined unit. As the child's mind grasps the relationship the defined and the standardized units must gradually be introduced to supplant the cruder notions.

HOW TO TEACH NUMBER

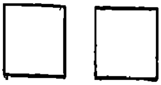
The theories of number and their psychological implications lead us to formulate four steps in teaching a number to a young child:

a. Present the whole quantity to be measured and thus give a vague idea of the number to be taught.
b. Select the unit or units of measure.
c. Perform the measurement process by counting the repetition of the unit.
d. Stress the equivalence between the whole quantity and the unit of measure. The ratio between them is the number.

1. In teaching the number *eight* we present eight sticks to the class and ask the pupils to reproduce with these what is made with chalk lines on the board. The first figure would be the following:



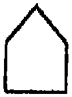
Every child first makes the octagon and then takes it apart to discover that eight one's are eight.

2. The next grouping,  presents two squares or boxes. The children are led to find out how many sticks in one box, in two boxes, and finally, how many four's make eight. The necessary drill and applications are now given on four and four, two four's, eight has how many four's, four and how many will give eight, had eight and lost four, etc. This second step (2) with its variety of drills and applications may take a number of periods, but time is really

gained, for the child is learning addition, subtraction, multiplication, and division combinations in the richest possible associations.

3. The children now set up Indian tents, Λ Λ Λ Λ and are led to discover that there are two sticks in one tent and in four tents, there are four two's or eight. The drill on 2×4, 4×2, 2+2+2+2, 6+2, 4+2+2, 2+6, 8−2, 8−4, 8−6, etc., must now be given, not with the symbols used here for economy of space, but in language forms similar to those in step 2. Here, again, the child is learning fundamentals in rich associations. This step alone may afford arithmetic material for a number of lessons.

4. The eight sticks are now used to make a pentagon and the child is led to discover that five sticks in the figure



and the three that remain, make eight. This relationship yields 5+3, 3+5, 8−5, 8−3, 5+?=8, 3+?=8, thus mastering combinations in related sequence rather than in isolated tables of addition and then subtraction.

5. The facts in the preceding steps are now subjected to a multiple sense drill, as follows:

a. Visual Appeal: Children point to pupils, seats, desks, books, window-panes, electric bulbs, etc., to show how many two's in eight, how many four's in eight, six and how many make eight, five and how many make eight, eight minus three will leave how many?

b. Auditory Appeal: Tap with pencil or ruler, ring bell, clap hands and children recognize the combination. The teacher taps xx xx xx xx and the child called on says "eight, four two's." The teacher taps xxxx xxxx and the pupil says "eight, two four's." All the combinations can be reviewed in this way.

c. Motor Appeal: Children take steps, hop, sing notes, arrange fellow pupils, tap with pencil, or clap hands to show their knowledge of any combination of eight.

> When the teacher asks, "How many two's in eight?" a pupil taps with pencil, xx xx xx xx, or arranges eight pupils who are standing in front of the class into four groups of two each. To the question, "Five and how many more make eight?" a pupil taps xxx, or arranges the eight fellow pupils into two groups of five and three and points to the group of three.

An analysis of the procedure outlined above shows that the lessons began with the entire quantity, eight; then the unit of measure, one or four or two, was selected; this was used to measure eight, and the final ratios or relations were discovered. The concept *eight* results from a variety of measurement processes in which the unit was not always one. When we consider that in the first grade most urban courses of study either make number work optional or prescribe little more than the four fundamental operations with numbers from 1 to 10 and counting from 1 to 100, there can be little reason for objecting to the length of time it would take to teach each of the numbers up to 12 by this procedure.

Effectiveness of teaching number through counting vs. through measurement. The distinctive elements in the method as outlined may be emphasized by comparing the number concept *eight* taught by a process of counting with that resulting from a method which emphasizes the measurement process.

Number Result of Counting	*Number the Result of Measurement*
1. Adopts 1 as the constant unit. Eight is taught as $7+1$. The new number is the result of adding 1 to the preceding number.	1. Any quantity is used as a unit of measure and a new number is evolved by measuring the quantity by the unit of measure selected
2. A mechanical method because all numbers are taught in same way. Counting gives number *a series meaning*.	2. Every change in unit gives a new combination and a new ratio. Measurement gives number *a group meaning*.
3. A synthetic process.	3. An analytical process.

Number Result of Counting	*Number the Result of Measurement*
4. Comparatively little mental activity is evoked in teaching eight as seven and one more; mere repetition will suffice.	4. Much mental activity is required to discover that eight is 4 2's, 2 4's, 5 and 3, etc. as well as 7 and 1.
5. To sustain interest new objects must be counted. Interest is thus directed to the sense material, not to the numbers and must therefore be characterized as fictitious.	5. Interest is maintained by changing the combination that produces eight, and leading the child to discover the identity among eight, 4 2's, and 2 4's. The same objects are used; interest is real, for it is resident in the new grouping, not in the changing objective material.
6. The combinations of a number, $6 + 2, 7 + 1, 5 + 3, 4 \times 2, 2 \times 4$, etc., are learned by pure rote through incessant repetitions.	6. Drill is necessary in any method, but here we have rational drill following the discovery of the combinations by the child

Steps in teaching any number through 12. Distribute the necessary objective material. Give as many of each object as the number to be taught. Children start with six colored sticks, six marbles, and six colored beads because number *six* is to be taught.

1. *The counting step.* Ask the children *to find out how many there are.* (*a*) First begin with things, marbles, boys, window-panes, chairs, pennies, toothpicks, colored splints, books—the things that are at hand in a classroom. (*b*) *Pictures* of flowers, birds, kittens, and people, are counted next. (*c*) Finally *symbolic number pictures* are used, lines, squares, dots, and rings.

In the initial step it is well to insist on ordinal counting. Thus, the child who is trying to find out how many children there are in the line, says, "The *first* is Mary; the *second,* Jane, . . . the *sixth,* Joan. There are six." The same procedure should be followed in finding out how many (counting) *pictures* and then how many *symbolic number* pictures there are.

2. *Measurement introduced.* The pupils are now required *to perform* with numbers. Here, clearly, the measurement process is introduced.

a. First have children pick out only six pupils, six books, and six chairs from a group decidedly greater than six. Again we begin with things which are then arranged to show two three's, three two's, four and two, five and one, and six one's. The lesson on number *eight* gives this procedure in greater detail.

b. The children are now asked to draw six oranges, six apples, and six boys in prescribed groupings—two rows of apples, three apples in each row. The units may be grouped to illustrate a story composed by a pupil or by the teacher. The story is told of the six birds who were so fond of one another that none of them ever flew alone. Sometimes they flew, two and two and two. [The children draw the following.]

Sometimes they flew three and three. [Now the children draw the following groups.]

c. More formal number material may now be used, blocks, splints, toothpicks. These are arranged by the children to show four and how many more make six; had six and lost two; how many three's in six, etc.

These three distinct series of measurement experiences should give the children ability to discover two three's, three two's, four and two, five and one, one-half of six, etc. While these number facts have not as yet been mastered, that is, habituated, each child ought to know how to find the answer to a problem like, "I had four marbles and I want six; how many more do I need?"

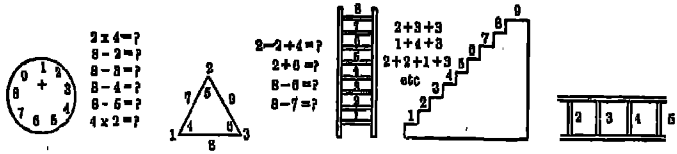
3. The symbol is now taught. The children are taught to recognize and to write 6 for six.

4. More formal steps are now taken to habituate the number facts of any number. This may be done as follows:

a. Multiple-sense appeal. Children tap four and ? equal six, thus, xxxx xx. They hop three and ? equal six, thus xxx xxx. Hand clapping and making lines, dots, and rings on blackboard or paper are other ways of having children express the number combinations.

The teacher may tap or ring a bell in such a way as to group the individual sounds, thus, xx xx xx. The children listen and when called upon say, "Six, three two's make six."

b. Variety of mechanical drills may be employed. These are circles, triangles, ladders, stairways, railroad ties, and the like.



The teacher, pointing to the figures 2 and 4, calls for the answer; pointing to 3, says, "and how many make," and then points to 6, etc.

c. The algorisms of fundamental operations:

$$\begin{array}{r} 2 \\ +4 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ -2 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ +? \\ \hline 4 \end{array} \qquad \begin{array}{r} 3 \\ \times 2 \\ \hline ? \end{array}$$

d. The equation form of drill:

$$2 \times 3 = ? \qquad 3 \times 2 = ? \qquad 4 + 2 = ?$$

e. The elliptical form:

$$2 \times ? = 6 \qquad 4 + ? = 6 \qquad ? + 3 = 6 \qquad 4 \times ? = 6$$

5. Employ the combinations in as many practical problems as possible and do not rely on unguided transference taking place to actual conditions of life. Although this is the

fifth step in our list it must be introduced much earlier in the learning process. While we are teaching Steps 3 and 4, practical problems of the following type must be given freely:

At 3 cents an apple, what will 2 apples cost?
At 2 cents an apple, how many can I buy for 6 cents?
I have 4 cents; how many more must I save to have 6 cents?

Moot Questions in Teaching Number

Shall number facts be taught through applications or as isolated forms? Some authorities insist that all number facts must be taught in relation to a practical situation to insure use of number knowledge in everyday life. The child who learns 3+3, 2×3, 4×3, and 4+3 as abstract facts by rigorous drill, reduces them to habit, we are told, but cannot apply them to solve a practical problem. Others insist that the forms must be mastered first, then practice must be given in making applications. This latter group approves rigorous oral, flash-card, and written drills beyond the learning point to insure overlearning and thus produce a safety zone for a measure of forgetting.

In his elaborate experimental study Washburne[2] found that, "Teaching the mechanics of arithmetic separately and then applying them to the solution of practical problems does not lead to difficulty in making practical applications of the mechanics of arithmetic to the solution of problems." However the superiority of this procedure over the other procedure was not very marked.[3] We must, therefore, wait for more experimentally derived data before we can speak with any degree of assurance on this question.

Theories of bonds and rationalization applied to learning facts. It is held by some that in learning arithmetic children must establish a host of new bonds, one for each number combination. The formation of two new bonds is required to learn 7+8 and 7+9. Since each of these bonds is quite independent of the other, it is asserted, no attempt should be made to make any rational association among these bonds. Do not teach 7+9 after 7+8 lest the child learn 7+9 by adding 1 to 7+8. 7+9 must be learned as independently of 7+8 as of 3+4. Drill on each bond assures mastery of that bond.

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[2] Carlton W. Washburne, "Comparison of Two Methods of Teaching Pupils to Apply the Mechanics of Arithmetic to the Solution of Problems," *Elementary School Journal*, Vol. 27, June, 1927, pp. 758–68.

[3] G. I. Buswell, "Summary of Arithmetic Investigations," Supplement No. 3 to Monograph 27, *Educational Monograph*, School of Education, University of Chicago, 1927, p. 13.

Another school of educational psychologists supports the theory of rationalization. Let each child, we are advised, discover each combination for himself before we supply drill or practice material. Lead the child to associate 7+8, 6+9, 8+7, 9+6, 5+10, and 10+5. As these are applied to practical situations, they are mastered. Drill should come only as a final stage to cap the preliminary thoughtful experiences and reactions to these combinations. Flash cards, games, mechanical aids, filling in figures in blanks—these are all valuable aids in drill and should come in the later, not in the preliminary, stages of learning.

These two theories are important because each determines the spirit of a lesson and the specific teaching procedure. No clear-cut selection can be made to-day because we lack accurate data, experimentally derived. Gestalt psychology makes out an excellent theoretical case against the bond theory; the behaviorists and the overzealous disciples of Thorndike are as effective in building up support for the bond theory. We know that children have learned number combinations when taught by methods advocated by both schools. We cannot assert the absolute superiority of the one over the other until we have reliable facts on which to base a decision.

Gradation of number for teaching. Investigators have sought to discover the comparative difficulty of numbers. The place of a number in the series from 3 to 12 is no index of its difficulty. Howell presents the following order of difficulty: 3, 4, 6, 5, 8, 10, 12, 9, 7, 11.[4] This is not the sequence that a teacher would expect. 6 seems less difficult than 5; 9 more difficult than 5; 9 more difficult than 12; the last three numbers are 9, 7, 11—all odd numbers.

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[4] Henry B. Howell, *A Foundational Study in the Pedagogy of Arithmetic* (Macmillan Co., 1914).

Brownell's study reaches other conclusions. There is little evidence that odd numbers are more difficult when visually perceived. In the lowest grades "Ease of apprehension is conditioned by the number of objects visually exposed."[5] Difficulty or ease of apprehension seems to be conditioned by the "quadrangle arrangement," that is, :: for 4, :: :: for 8 and :: ∴ for 7, are all more easily grasped than . . . . for 4, ... for 8, and . . . . for 7. From this fact it would seem that even numbers would be easier to learn when expressed by a quadrangle arrangement.

Out of the mass of conflicting data no clear-cut conclusion emerges. The entire problem is perhaps a matter of no great moment since equal mastery of all numbers up to 12 must be developed.

The use of number pictures. Not all teachers are convinced that number pictures are useful. The number pictures do not give rise to a number concept because more than mere contact with multiplicity is necessary before number ideas develop. Without measurement processes no adequate idea of number is conceived.

Number pictures do present some sense experiences out of which number relations may emerge. Investigations by Lay tend to show that most children prefer a visual appeal to an auditory appeal, even when the sounds are heard in rhythm. Hence number pictures make a visual appeal to a child who reacts favorably to this sense material. There are arrangements of elements in number pictures, which tend to facilitate or retard the learning of number:

1. One row pictures are not to be favored; for example, o, o o, o o o, o o o o, o o o o o, ...

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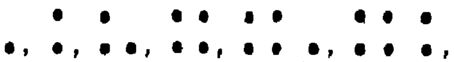
[5] William A. Brownell, "Development of Children's Number Ideas in Primary Grades," *Supplementary Educational Monographs,* No. 35, School of Education, University of Chicago, August, 1928, p. 18.

2. Any arbitrary or accidental arrangement is not effective; for example, $\begin{matrix} & \circ\circ\circ & \\ \circ & & \circ \\ & \circ & \end{matrix}$ for 6

3. In the best arrangement elements are grouped in two rows, in quadratic form, with equal spaces; for example,



4. The circle of about 5 mm. is superior to the dot, line, or other mark.

5. The dark circle on a light background or light circle on a dark background seems to give the best results. Card and blackboard presentation are therefore effective.

Although these conclusions are not final, they are supported by data obtained in investigations planned and carried out with great care.[6]

THE TEACHING OF COUNTING

Its history. Counting is not a natural process; it is laboriously developed to meet the needs of life. Untaught people do not know how to count, but use such aids as the following: finger and toes; known objects—as many as legs of a horse; made objects, like notched sticks, beads on wire similar to an abacus, pebbles in a bag to correspond to a given number (as many pebbles as sheep). The etymology of the words *calculate* and *calculus* is found in words which meant *pebbles.* Our present number system is borrowed from the Orient and seems to imply failure on the part of Europeans to evolve an adequate number system. The decimal notation is the cumulative product of ages of thinking about number and quantitative relations.

Counting defined. Counting is a process of successive "inner acts, which are, in essence, inner articulations of number names."[7] Experiments were made by Judd upon adults who were asked to count tappings, flashes, and musical sounds. The results abound in errors and reveal marked variations. Some count flashes better than tappings; others prefer the auditory experiences. Some make few errors; others make many. Adults, when asked to make marks on paper by tapping for ten seconds, make about as many marks as they can count orally or silently in that time. "The rate at which one can send motor impulses to the hand is very closely related to the rate at which one can send simple motor impulses to the vocal chords."[8] The conclusions are obvious: counting is an acquired complex of skills and involves mental activity of a high order.

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[6] Howell, *op. cit.*; Brownell, *op. cit.*

Psychology of counting. In its origin, counting is probably a motor response to successive experiences prompted by an "inner rhythmic sense," but without absolute reference to external objects. In the early stages counting is accompanied by motor responses like head or finger movements, finger or foot tappings, body swayings, or touching successive objects. Counting may be regarded as an outward manifestation of inner rhythmic impulses. We referred, in an earlier connection, to the practice, which young children have, of counting without reference to any objects. A child who knows the number names from one to ten will count ten marbles when he has only six. He keeps saying the number names until he has reached the last number name which he knows, pointing with each name to a marble. But the last number name may bear no relation to the quantity he is counting.

Application of the psychology of counting to teaching. Three distinct steps are therefore necessary in teaching children rational, not rote, counting:

1. The acquisition and control of a series of numbers names.
2. The recognition of the identity of the individual elements in a group.

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[7] Charles H. Judd, "Psychological Analysis of Fundamentals of Arithmetic," *Supplementary Educational Monographs*, No. 32, University of Chicago, February, 1927, p. 32.

[8] *Ibid.*, p. 3.

3. The making of a one to one correspondence between object and name.

Ability to count must not be taken for granted despite the glib recital of number names by children. Lessons in counting must be graded to insure mastery of each of the steps listed above. In the lesson on number eight the reader will see these steps illustrated.

Gradation of counting exercises. No gradation in counting is acceptable to all because we have no scientific data upon which to base a specific plan of gradation. We must rely on experience and the logic which seems inherent in the successive steps of unfolding subject-matter. Here are various gradations suggested for counting.

Plan I

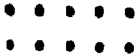
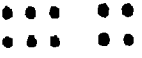
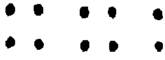
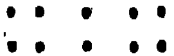
1. Counting 1–25 by 1's with and then without objects
2. " 1–50 by 1's and 2's with objects
3. " 1–50 by 1's, 2's, 5's, and 10's
4. " 1–100 by 1's, 2's, 5's, and 10's
5. " 100–200, 200–300 by 10's
6. " 100–1000 by 100's
7. " beyond 1000 by 100's as child learns to read and write large numbers.

Plan II

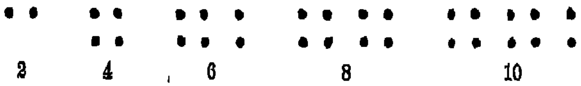
Follow the general outline of *Plan I* but add counting by 3's, by 4's, 6's, 7's, 8's, 9's, 11's, and 12's as multiplication is introduced.

Counting by 2's or by any other number is graded as follows:

1. Counting by 2's, begin with zero, go through 24.
2. Counting by 2's, begin with a multiple of 2, go through 24
3. Counting by 2's, begin with a multiple of 2, but in reverse order.
4. Counting by 2's, begin with any number, thus 5, 7, 9 . . . 17, 19, 21.
5. Counting by 2's, begin with any number, but in reverse order, 19, 17, 15.

This form of counting is sponsored as ardently by some as it is condemned by others. Those who recommend this practice ascribe many values to it. (*a*) Practice in rapid addition and subtraction is assured. (*b*) Graded steps 1, 2, and 3 of above give the child mastery of the products in the multiplication tables. (*c*) The child becomes familiar with the sequence of numbers and the place of each in any limited range. (*d*) "Counting by 2's, beginning with 1 and with 2, counting by 3's beginning with 1, with 2, with 3, and with 4, and so on, make easy beginnings in the formation of the decade connections.[9] (*e*) The mind is unable to deal directly with large groups. When confronted by a group like , it regroups the elements into  or  or . The implications are clear: teach number by groupings and give drill in counting by 2's, 4's, 3's, etc.

But those who reject this form of counting are very emphatic in their opposition. They argue: (*a*) Adequate drill in rapid addition and subtraction can be given without these forms of counting, which do not arise in practical situations. (*b*) Through these forms of counting children learn the products in multiplication tables without the factors of that product, that is, in 3, 6, 9, 12, 15, 18, *18* is learned without 3×6, or 2×9, and is therefore really not a multiplicative fact. Children soon learn to count to a product, 4, 8, 12, 16, 20, *24* instead of mastering 6×4 or 3×8 or 2×12=24. Experience often reveals that children acquire a series of number names which are little more than sounds. Instead of 2, 4, 6, 8, they may say, *intry, mintry, cutery, corn* with as little stimulation of the sense of number. When shown : : : :, they arrive at 8, all too frequently, by counting by 1's—not by counting by 2's or by 4's. Hence there must be longer teaching of counting through concrete and semi-concrete experiences. Let children count, thus:



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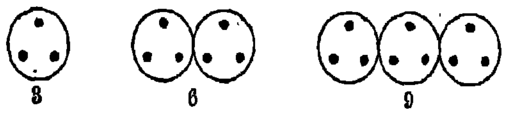
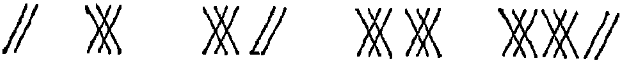
[9] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), pp. 75–76.



Or, as they count, 2, 4, 6, 8, 10, they draw lines as follows:



Brownell's[10] study shows clearly that when children master number names they do not necessarily acquire corresponding number ideas and that, for a long time, names are not translated into ideas.

The debate may become protracted but it will prove nothing conclusively. Most authorities sanction only a little of this type of counting, say by 2's to 50, by 5's to 100, and by 10's to 500. Counting by 3's or 6's or 4's they are likely to regard as skills of relatively little value acquired at an inordinate cost in time. Psychologically, this position is entitled to more than passing respect because each type of counting involves a mental pattern all its own. To acquire all these forms of counting one must acquire cumulatively and progressively a set of number patterns at a time when more useful techniques demand mastery.

What should counting give the child? Systematic exercise in counting should give the child a reliable foundation for his later number work. The specific gains seem to be at least five: (1) the number names in proper order; (2) the rhythm of the decimal system; (3) the relative location of any number in the number series; (4) a first step in understanding number; (5) a preparation for the fundamental operations.

Reading and writing numbers. *The figures 1–9.* Children learn to read figures 1–9 and 0 incidentally, in the course of all lessons. The writing of these must be taught as an arbitrary process by an imitative method with repetitions that make a multiple-sense appeal. To teach the figure 7, the following steps may help:

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[10] William A. Brownell, "Development of Children's Number Ideas in the Primary Grades," *Supplementary Educational Monographs*, No. 35, University of Chicago, November, 1928, Ch. iii.

1. Teacher writes a series of 7's on blackboard and explains the two characteristic lines in 7, namely, ‾ and /.
2. The children now imitate by making large 7's in the air. Large free-arm movement to impress direction and slope must be encouraged.
3. Visualize the 7. Children close eyes and attempt to "see" the 7, writing the figure in the air or with finger on the desk as the 7 is visualized.
4. Children now pretend to write 7 on an imaginary paper. The teacher corrects and urges smaller forms until the desirable size is approximated.
5. Crayon and paper are then used and the 7 is attempted. Those who produce incorrect figures are sent to the board to trace the 7's there; the others are corrected and encouraged to repeat the writing of the 7 until habit is formed.

Figures of two and three orders. Patience is the only requisite for teaching children to write any digit. The difficulty in figure-writing is not experienced until 11, 12, 14, and 21 must be written. If the symbol # represented eleven, ϕ represented twelve, and ¶ represented thirteen, there would be no new learning difficulty, because eleven, twelve and thirteen are new ideas and are represented by new symbols, #, ϕ, and ¶, respectively. But when 11 is eleven, 12 is twelve, and 13 is thirteen, the child finds old symbols representing new ideas. In attempting to establish, for the child, an association between new number ideas and new arrangement of known symbols, we find the key to the difficulty experienced by pupils. To meet the pedagogical challenge teachers devised the plan of converting 13, 14, or 19 sticks into 1 bundle of 10 sticks, and 3, or 4, or 9 sticks over. The same was done with beads. Children who were given 13 beads put ten into a small bag and said, "Thirteen is 10 and 3 over." They were then taught to write 13, 14, 15, 16 beads, thus,

Bag	Beads
1	3
1	4

and later in closer proximity, 13, 14, 15, and 16.

While the use of sticks or beads may be necessary in foreign schools, the method is needlessly artificial for American children who know the dime and cent relation. The first step, therefore, is to give oral practice in changing cents into dimes and cents; 15, 16, 18, 12, 14 cents are changed into 1 dime and 5, 6, 8, 2 or 4 cents, respectively. The children then write dimes in one column and cents in another. A small silver circle heads the dime column on the blackboard, and a copper-colored circle, the cent column; 15 or 16 is therefore written as 1 under dimes and 5 or 6 under cents. For a short time, numbers 10 through 19 are reduced to dimes and cents and are written as such. But, as the days pass, the signs of the tens' and the units' place are eliminated and no reference is made to dimes and cents. The child acquires the habit of writing 15, 1, 14, regardless of the name of the objects. Numbers of three orders may be taught by using dollar, dime, and cent equivalents.

Drill must be given in writing numbers that are dictated, in reading numbers from arithmetic books or charts, and in writing, in figures, numbers that are spelled out in the textbook or on a chart. The aim must be to develop accuracy and instantaneous recognition or visualization of numbers.

Reading and writing of large numbers. Teachers are frequently urged to limit number-reading and -writing to small numbers, not to exceed ten thousands. We must not carry the movement for the simplification of arithmetic too far. Children are interested in large numbers. They keep asking, "What comes after hundred thousands? after millions? how do you write millions?" This curiosity should not be discouraged because large numbers are very common to-day. Newspapers and magazine articles discuss budgets, population, relief funds, exports, and imports in terms of dollars or of bulk. The cost of military preparation or the extent of comparative national armaments is presented in tragically large numbers. There is no reason why all citizens, even those classed as dull normals, should not be able to read such numbers and have some understanding of their magnitude.

Correctness of form. It is well to insist on correct form in reading and writing numbers from the very beginning. Commas after each third place, thus, 750,892 or 1,054,782 are not correct and should not be taught. Neither should the introduction of the conjunction *and* be permitted in reading whole numbers. "Two hundred *and* fifty-seven" is incorrect for 257; the correct form is "two hundred fifty-seven." Similarly, "five thousand *and* three hundred *and* eighty-five" is incorrect for 5385; the correct form is "five thousand three hundred eighty-five." The conjunction *and* is limited to mixed numbers with common or decimal fractions. To illustrate,

759.34 is read, seven hundred fifty-nine *and* thirty-four hundredths.

8672.034 is read, eight thousand six hundred seventy-two *and* thirty-four thousandths.

45.546 is read, forty-five *and* five hundred forty-six thousandths.

575.1436 is read, five hundred seventy-five *and* one thousand four hundred thirty-six ten thousandths.

$1210.57 is read, one thousand two hundred ten dollars' *and* fifty-seven cents.

Since there is only one correct form no other form should be tolerated. Then the child will be spared the task of unlearning wrong responses.

Roman numbers. Roman notation must be taught in the elementary school because it is commonly used, especially in decorative schemes, in dates, in money, and in governmental documents. That which is in common social use has a legitimate claim for inclusion in the curriculum.

Incidental teaching in the early grades. Roman numerals, I-XII, should be introduced early—in the latter part of the second grade—and taught incidentally. Use Roman numbers to supplant the Arabic forms in numbering the blackboard panels and the boxes or lines in a game. In the early lessons on telling time a dial with Arabic numerals is used. After the children have made progress in telling time, substitute a dial with Roman numerals. Many children do not question the substitution. They accept the new symbols, III and IV, and associate them with the place formerly occupied by 3 and 4. With hardly a word of explanation, the children learn the first twelve Roman numerals. This information is adequate to meet all needs in the first three or four grades.

Later development of Roman notation. In the fourth or the fifth grade we may begin to carry the Roman notation to L. The instruction should now be made more systematic. From the numerals I-XII, derive, inductively, the three simple rules that govern the Roman notational system:

1. Repeating a symbol repeats the value; for example, one I is 1, two I's represent 2 and three I's represent 3.
2. A smaller number before a larger number indicates subtraction; for example, I before V means $5-1$, hence 4; I before X means $10-1$, hence 9.
3. A small number following a larger number indicates addition; for example, V followed by I or II or III means $5+1$ or $5+2$ or $5+3$, hence 6, 7, or 8; so, too, X and I represent 11, X and II, 12.

Each day the system is carried a little further. One day from XII to XV; another, from XVI to XIX; a third from XX to XXV. At no time is a whole period devoted to this work. Three to five minutes a day will suffice to meet any reasonable course-of-study assignment. In the succeeding semester the Roman numbers may be carried to C; in a third semester, to D; in a fourth, to M; in a fifth, to any number. Presented in this gradual way Roman numerals are learned with little effort and less concern.

Throughout the school course in arithmetic occasional review of Roman numerals should be provided. Children should be asked to read Roman numbers and to justify their answers; to write Roman numbers from dictation; and to write, in Roman form, numbers that are spelled out. There can be no justification, however, for requiring children to perform fundamental operations with Roman numbers.

PROBLEMS FOR STUDY AND DISCUSSION

1. For each of the following statements give supporting data and an illustration:

(*a*) Pestalozzi would have approved the teaching of number names and then associating them with objects.
(*b*) Pestalozzi was a formal disciplinist in the teaching of arithmetic.
(*c*) Many children are eager to learn counting because it is rhythmic.
(*d*) Most children are eager to learn counting; they recognize its social value.
(*e*) Pedagogical application can be made of the fact that the first concept of number is ordinal, not cardinal.
(*f*) Number is not a mental picture which corresponds to reality.
(*g*) Learning the number idea is essentially a process of grouping experiences.
(*h*) A unit and unity are not identical.
(*i*) Ratio analysis can be acquired by young children.
(*j*) No adequate number concept can be acquired merely by counting.
(*k*) Counting gives a rhythmic sense of spatial and temporal relations; measurement gives a sense of equivalents.
(*l*) In teaching a number begin with a quantity which that number represents.
(*m*) Number is taught by an analytic-synthetic, not by a synthetic-analytic process.
(*n*) Mastery of number combinations does not insure ability to use these combinations in natural situations.
(*o*) Gestalt psychology is hostile to Thorndike's theory of bonds.
(*p*) Visual appeals are more effective than auditory appeals in early number teaching.

2. List three important questions concerned with the teaching of number that cannot be answered with any degree of finality until we have adequate data experimentally derived.

3. Number pictures do not give rise to number ideas, because more than mere contact with multiplicity is necessary to acquire a number concept. Give reasons with illustrations. What more is needed for the acquisition of a number concept?

4. Counting and measurement may be distinguished, but in actual teaching, the two fuse so that they appear like two aspects of the same activity. Discuss with illustrations.

5. Distinguish rational counting from rote counting; distinguish teaching counting from teaching number.

6. Analyze the steps in teaching the number nine as a group of ratios, and show that by your procedure the child would acquire the five skills resulting from the systematic teaching of counting.

7. Describe the religious significance associated with numbers by primitive and modern people.

8. Describe a system of notation other than the Arabic or the Roman.

9. Trace the history of the zero and the recognition of the significance of "placing" in a system of notation.

CHAPTER XX

TEACHING ADDITION OF WHOLE NUMBERS

Oral Addition

Incidental addition. The child who learns his numbers through a process of measurement rather than through counting has learned to add and to subtract. Reference to the lessons on numbers six and eight[1] reveals the fact that in learning number six the child has performed the following additions: 3+3, 2+2+2, 4+2, and 5+1. By the time number twelve is learned, the child has a very large body of number facts and skills about addition.

The basic combinations in addition. *What are they?* The sums, never greater than 18, obtained by adding any two digits make up the basic combinations. Mathematically, there are only forty-five combinations because 5+4 and 4+5 are really one and 5+0 and 0+5 are not combinations since the positive quantity remains unchanged. Psychologically, there are 100 combinations because 4+5 and 5+4 are not regarded as the same combination. A child may know the one and not the other. So, too, 5+0 is a real addition fact for the child. These combinations, when mastered, give the child the basic skills for successful addition. In the last analysis, all addition is an application of these combinations.

A simple device for obtaining the forty-five combinations is found in Fig. 5 (reverse combinations and zero combinations are not included):

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[1] See pages 235–238.

	1	2	3	4	5	6	7	8	9	
1	2	3	4	5	6	7	8	9	10	9 Combinations
2		4	5	6	7	8	9	10	11	8 "
3			6	7	8	9	10	11	12	7 "
4				8	9	10	11	12	13	6 "
5					10	11	12	13	14	5 "
6						12	13	14	15	4 "
7							14	15	16	3 "
8								16	17	2 "
9									18	1 Combination
										45 Combinations

FIG. 5.

To obtain the first set of combinations add 1 of the vertical column to each of the numbers in the horizontal line; this gives 1+1=2, 1+2=3, 1+3=4, 1+4=5, . . . 1+9=10; to obtain the second group of combinations add 2 of the vertical column to each number in the horizontal line, thus, 2+1=3, but since 1+2 is found in the preceding line, this is omitted: 2+2=4, 2+3=5, . . . 2+9=11.

The basic combinations of addition are designated in three different ways: The first refers to them as the *forty-five* combinations. 5+4 and 4+5 are regarded as one combination. Knowing the one, it is argued, the child knows the other. Here we find unlimited faith in the transference of knowledge of number facts.

The second designation refers to them as the *eighty-one* combinations. 4+5 and 5+4 are counted as two combinations; it is believed that a child may know the one without knowing the other. The forty-five combinations do not become ninety because there are nine doubles (1+1; 2+2; 3+3 . . . 7+7 . . . 9+9) which cannot have reverse combinations. The zero combinations are not taken into consideration because it is believed that they are too obvious to teach.

In the third designation, these number facts are referred to as *the one hundred combinations.* To the eighty-one combinations are added nineteen zero combinations (0+1, 0+0, 2+0, 0+2, etc.) because, it is held, that 2+0 has as much identity as any combination without a zero and requires as much learning as 5+7.

Gradation of the forty-five or 100 combinations. Much laborious research has been carried on to discover the relative difficulty of each one of these combinations. 1+1 seems the easiest, 2+2 and 5+5 are among the easiest, but 9+8 and 8+9 seem to be among the most difficult for the learner. Between these extremes there must be varying degrees of difficulty, the determination of which has been a challenge to many investigators.

The additive combinations have been classified according to the number of errors made by children. The difficulty of a combination is determined by the number of children who err in it. A second mode of measuring difficulty is based on reaction time. The speed with which the answer to a combination is given determines its degree of difficulty. The time element is considered by many very important, because if time is not limited, the child may obtain an answer by counting. Thus, the sum of 5 and 7 may be obtained by counting 5, 6, 7, 8, 9, 10, 11, 12 or by seeing 5 in the 7 and then adding 5+5+2, 12. A child does not really know a combination unless he recognizes it instantly—and, of course, without computational aids. Clapp[2] used the first method (number-of-errors) in studying the relative difficulty of these additive combinations, and Smith[3] used the second (reaction-time). While investigators do not agree in their findings, it is interesting to note that all the combinations except three in the most difficult third of Smith's list appear in the most difficult third of Clapp's list.[4]

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[2] F. L. Clapp, *The Number Combinations: Their Relative Difficulty and the Frequency of their Appearance in Textbooks.* Bureau of Educational Research, Bulletin No. 1 (University of Wisconsin, 1924), pp. 20, 126.

[3] J. H. Smith, "Arithmetical Combinations," *Elementary School Journal,* Vol. 21, June, 1921, pp. 762–770.

The practical value of these researches seems very slight. The implication of a graded list is that we teach the items in the order set forth. But we may not be warranted in drawing such an inference. May not the alternation of a simple and a difficult combination facilitate the learning of the combinations? The former may renew the child's faith in his capacity to learn, and thus enable him to meet the difficult combination with a very favorable mind-set. Then, too, the difficulty of a number combination depends upon many conditions: Is it met in isolation or in a practical problem? Is the combination presented in a timed or in an untimed test? Is the combination tested with children in Grade 3 or in Grade 6?

A few definite conclusions, however, may be drawn from these inquiries[5] into the relative difficulty of the additive combinations:

a. 8 + 5 and 5 + 8 are not of equal difficulty; adding the smaller number to the larger is easier, hence 8 + 5 should precede 5 + 8.

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[4] Clapp gives the following order of difficulty beginning with the most troublesome combinations and proceeding to the least difficult one.

8 + 5
7 + 9
5 + 8
9 + 7
6 + 8
6 + 9
5 + 7
7 + 8
8 + 7
9 + 6
5 + 9
8 + 9
8 + 6
4 + 7
7 + 5
7 + 5
4 + 9
9 + 5
9 + 4
6 + 7
5 + 6
4 + 6
7 + 6
7 + 4
9 + 8
3 + 7
9 + 0
2 + 6
9 + 3
0 + 6
6 + 5
3 + 8
3 + 4
3 + 9
2 + 3
3 + 5
6 + 3
7 + 3
2 + 7
8 + 4
4 + 8
8 + 0
1 + 0
5 + 2
4 + 2
1 + 2
5 + 3
0 + 3
0 + 5
5 + 1
7 + 0
8 + 7
0 + 1
7 + 2
1 + 9
0 + 5
8 + 1
6 + 2
0 + 4
3 + 6
0 + 2
2 + 4
3 + 0
4 + 5
0 + 8
6 + 0
8 + 3
8 + 2
6 + 4
1 + 4
9 + 1
5 + 0
6 + 6
3 + 2
4 + 3
1 + 5
7 + 1
2 + 9
2 + 5
2 + 8
4 + 4
4 + 1
3 + 1
4 + 0
1 + 8
9 + 9
7 + 7
2 + 0
6 + 1
5 + 4
3 + 3
1 + 1
9 + 2
8 + 8
1 + 3
1 + 6
1 + 7
2 + 1
2 + 2
5 + 5
0 + 0

[5] Further references on gradation of the combinations in addition are:

Heilman and Shultis, "A Study in Addition," *Research Bulletin* No. 1, State Teachers College, Greeley, Colorado.

B. R. Buckingham, "Arithmetic at the Crossroads," Ohio State University *Educational Research Bulletin*, Vol. 6, January 5, 1927, p. 2.

b. For teaching purposes there are 100, not 45, combinations.
c. Successive combinations that have a common element, thus, 3 + 6, 3 + 7, 3 + 8, should be avoided, because children may derive the later ones from the earlier. 3 + 7 may become, "3 + 6 = 9, now add 1, 10."
d. The larger the numbers, the more difficult is the combination.
e. The doubles are comparatively simple.

Teaching the basic combinations of addition. Older methodology worked out a seemingly logical system of teaching number facts: first the formula, then the application; first 7+5=12, then, "Had 7 cents and earned 5 cents; how much did I have?" The justification was that one cannot apply what one doesn't know. To-day we begin with practical problems and derive the formula from them: first, "I had 7 cents and earned 5 cents," then 7+5=? Motivation is thus assured and understanding precedes the establishment of a specific bond.

Let us turn to a suggested procedure for teaching the combinations.

1. Concentrate on one combination at a time, say 8+7, and teach all the facts about it.[6]

2. Begin with concrete materials and the entire group, that

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[6] Courtis graded the 100 combinations as follows, on the basis of studies he made some years ago:

Group I.	Very Easy	0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8 9	1 2 and the reverse 0, 0, etc.
Group II.	Easy	1 1 1 1 1 1 1 1 2 2 3 4 5 2 3 4 5 6 7 8 9 3 2 3 4 5	and the reverse,
Group III.	Average	2 2 2 2 2 3 3 3 3 3 4 2 3 4 5 6 7 2 4 5 6 7 5 7	and the reverse,
Group IV.	Hard	2 2 3 3 4 4 4 4 6 7 8 9 8 9 8 9 6 7 8 9 6 7 8 9	and the reverse,
Group V.	Very Hard	5 5 5 5 6 6 6 7 7 8 6 7 8 9 7 8 9 8 9 9	and the reverse.

is, with fifteen books or fifteen children. "How many children (or books) have we in this group?" asks the teacher, pointing to the fifteen children. The fifteen children form one group as they are counted by a classmate. We must note that the group is not counted to discover the sum of eight and seven. That would be very undesirable for it would teach children to rely on counting when they need the sum of two numbers.

3. The teacher then regroups the original group into two smaller groups, one of eight and the other of seven. The fifteen is now analyzed into eight and seven. "How many children did you say there were altogether? How many here? How many there? Then, eight children and seven children are how many children?"

Prohibit all counting at this stage to ascertain *fifteen.*

4. Repeat this procedure with other concrete material—fifteen marbles, fifteen pencils, fifteen pads, and fifteen chalk lines on the blackboard. Again, make sure that the whole group is counted *before* the subgroups. Attention is directed to the two smaller groups after the size of the original large group has been ascertained.

5. Present the algorism for addition by writing $\begin{array}{r}8\\ \underline{7}\\ 15\end{array}$ and saying, "eight and seven are fifteen." Pupils write this as they say "eight and seven are fifteen."

6. Repeat step 5 with the reverse of the combination, that is, with 7+8. Experimental studies reveal the superiority of teaching a combination and its reverse "together" (that is, in the same lesson) over the teaching of each of the two forms without reference to the other.[7]

7. Introduce the drill with perception cards. A cardboard of suitable size reveals $\begin{array}{r}8\\ \underline{7}\\ ?\end{array}$ on one side, and $\begin{array}{r}7\\ \underline{8}\\ ?\end{array}$ on the other.

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[7] B. R. Buckingham, "The Social Value of Arithmetic," *The Twenty-Ninth Yearbook* of the National Society for the Study of Education (Public School Publishing Co., 1930), p. 34.

If a pupil fails to give the correct answer when a card is shown to him, the teacher should call on another child. Within half a minute the child who failed should be called again to give the answer to the combination he missed. Each day new perception cards are added for the new combinations. Periodically, old perception cards are taken out after tests show that certain combinations are adequately learned.

8. Teach the subtraction facts based on eight, seven, and fifteen. Using concrete material, ask, "seven and ? make fifteen? What and eight make fifteen?" The forms, *from, take, minus, left, subtract* may well be postponed until all the 100 combinations have been taught.

9. The equation form of drill may now be introduced:

$$8+7=?\qquad 7+8=?\qquad ?+7=15\qquad 15=8+?$$

These forms may be presented on perception cards. The symbols are explained when they first appear.

10. Apply the number facts, as they are learned in each of the above steps, to as many practical problems as possible so that the additive idea is thoroughly understood. *It is most unwise to wait for the practical applications until the last step is reached.*

As progress is made in the combinations of addition, children take the steps outlined above quickly and with less direction by the teacher. In actual practice the entire lesson on a new combination takes only a small part of a twenty-minute period in number work.

The zero combinations. There is no value in teaching the zero combinations by a procedure suggested for the others. The need for zero combinations does not arise in dealing with concrete materials. They enter into calculations only in written work when the addends contain zeros, as in examples like, 30+7+104+162. The addition of zero should be dealt with when the situation arises. No carefully planned series of teaching steps is necessary for zero combinations.

Games. While games designed to give practice or drill are generally sanctioned, they must nevertheless be evaluated critically. The time spent on these games may be out of keeping with the value of the contribution they make to a lesson. We have yet to prove that they are more effective than rational motivation followed by functional drill. In the absence of objectively derived data, no final judgment can be made.

Games must be simple, and attention must be focused on number combinations rather than on the process of the game. The primary aim of the number game is to teach number facts. This objective must ever be kept in sight. Nor is it advisable to stimulate the competitive spirit too much in these games. With a little ingenuity teachers can devise suitable number games, and adapt to number work the existing games used in other school subjects.[8]

Objectives to be attained in teaching number combinations. The following questions indicate the factors that determine the success of a series of lessons on number combinations:

1. Is the drill motivated and made functional?
2. Is the recognition made instantaneously and without counting?
3. Are many children called per minute?
4. Is there a "calculated frequency of appearance" of each combination?
5. Are these drills given in short but frequent periods?
6. Are there periodic tests to discover the specific combinations that each child has not yet mastered?
7. Are drills individualized to meet the weaknesses of individual pupils?

Carrying the basic combinations to higher decades. Unless the basic combinations in addition are carried beyond the sum of 18, they will not give the child an adequate set of skills for every kind of useful addition. Adults as well as children who recognize instantly 6+7, hesitate at 56+7 or 76+7. The tendency seems to be to say for 56+7, "sixty, sixty, sixty-three" and for 76+7, "eighty, eighty, eighty-three." The 3 never changes when a 6 and a 7 are added; yet the hesitation is on the fixed element.

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[8] Any good present-day textbook in arithmetic contains many suitable games.

The explanation is simple: skill developed in the basic combinations is not readily transferred to higher decades. We must provide adequate practice in making this transfer. Carrying the combinations to higher decades is sometimes referred to as "upper-decade addition," or "adding the endings."

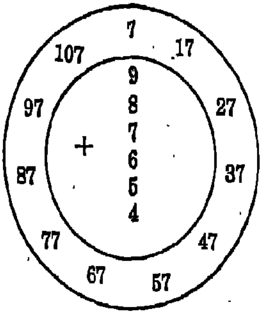
There are two classes of upper-decade additions: In the first, there is no bridging of tens; 34+5, 42+6 are examples. In the second, the addition brings the sum to one decade higher than the larger addend; 34+7 and 42+9 are illustrations. The object of this classification is to indicate an inevitable gradation.

Upper-decade addition is very important but it need not be presented as a set of new combinations. Experience proves clearly that it can be taught adequately as an extension of corresponding combinations in the basic combinations. After the child has mastered 8+7 and 7+8, he should be taught 17+8 and 57+8. The blackboard or chart work may take on the following appearance:

$$\begin{array}{rrrrrrrrr} 7 & 17 & 57 & 27 & 37 & 87 & 67 & 77 & 47 \\ \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} \\ 15 & & & & & & & & \end{array}$$

$$\begin{array}{rrrrrrrrr} 8 & 18 & 58 & 28 & 38 & 88 & 68 & 78 & 48 \\ \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} \\ 15 & & & & & & & & \end{array}$$

Another device for ready drill in carrying the basic combinations to higher decades consists of two concentric circles, thus,



Add numbers in inner circle to the numbers in outer circle as rapidly as the teacher points to them; e.g.,

7+5, 7+55, 7+85, etc., 9+5, 9+15, 9+25, 9+45, etc.

The numbers in the outer circle may be erased and 6, 16, 26, 36, or 9, 19, 29, 39, etc., substituted. The mechanical devices suggested for the basic combinations can easily be modified for our immediate purpose.

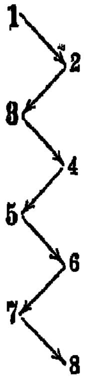
Short, frequent periods devoted to this type of drill bring gratifying results and tend to generalize the basic combinations of additions.

Addition through counting exercises. In the preceding chapter we discussed fully the values of counting by 2's, 3's, 4's, 5's, etc., as preparatory exercise for addition and multiplication. Those who sanction such counting teach it by the following practices:

1. Count 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, etc.
2. Count 1, **2**, 3, **4**, 5, **6**, 7, **8**, 9, **10**, 11, **12**, 13, **14**, etc.
   The teacher writes these two lines on the board. The children read the second line, emphasizing the even numbers, thus 1 (soft,) 2 (loud), 3 (soft), 4 (loud), 5 (soft), 6 (loud).
   Repeat a number of times until 2, 4, 6, 8, etc., have made strong visual and auditory appeals.
3. Count 2, 4, 6, 8, 10, 12, erasing odd numbers. At first, taps, or arm and finger movements may accompany the count; later these are omitted. Experiments made in these counting forms show that the introduction of rhythmic quality and auxiliary movement of finger, arm, or foot will aid in the acquisition of the new series.
4. Similarly (*a*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, etc.

(*b*) 1, 2, 3, 4, 5,
loud soft loud soft loud
6, 7, 8, 9, 10,
soft loud soft loud soft
11, 12, 13
loud soft loud

(*c*) 1 3 5 7 9 11 13 etc.



1	2
3	4
5	6
7	8
9	10
11	12 etc.

Similar procedures are used in teaching counting by adding by 3's, 4's, 5's, etc. It is important to remember that when the series, 1, 3, 5, 7 . . . or 2, 4, 6, 8, 10 . . . is recited by rote, the responses have diminished value because the child does not perform rapid successive additions. When this stage is reached, the assignment should be made by indicating an initial number higher than 1. Thus, "Count by 2's, begin with 7; by 3's, begin with 9." Now the child cannot fall into a rhythm which requires no arithmetical thinking.

WRITTEN ADDITION

Written addition to be delayed. In the final analysis, proficiency in written addition is conditioned upon the mastery of the 100 basic combinations and of upper-decade addition—two elements that make up the oral work in addition in the introductory grades. Nothing is gained by introducing written addition too early. In fact, much is lost, because inadequate preparation for written addition leads to many bad habits; slow responses, finger calculations, and excessive "talk"—all arising from the pupils' lack of confidence and imperfect habituation.

Gradation of written addition.

1. No Carrying.
    a. Single column
    b. Two columns
    c. Three columns
2. Carrying

a	*b*	*c*	*d*	*e*	*f*	*g*
Higher decade additions	Carrying in one column	Carrying in two columns	Carrying in three columns	Same as *d* with zeros	Addends with irregular number of places	Any combination of preceding steps with increase in the number of places and of addends
9 7	19 7 8 6	19 27 38 46	249 526 174	209 526 107	846 82 106 40 8	
49 7						
9 7 8						

Single-column addition. Adding a single column of numbers involves nothing new. It does, however, open many possibilities of acquiring bad habits of thinking and working. Against these we must bend all our energies.

Discourage talking. In finding the sum of the single column to the right, children should be trained to think: 9, 16, 21, 29, 35, 38. They should never be permitted to say to themselves: 9 and 7 are 16; 16 and 5 are 21; 21 and 8 are 29; 29 and 6 are 35; 35 and 3 are 38. The answer is 38. This excessive inner talk retards rate and diffuses attention. A new source of errors is thus developed. Children in the upper grades who *talk* during their addition find that they make more errors when they try to eliminate this excessive inner speech. This is to be expected because we are interfering with their addition habits, undesirable habits to be sure, but habits none the less. In the early grades, when single-column addition is first introduced, children must be led to acquire the habit of rapid addition without vocalizing each of the successive addends and all the partial sums.

$$\begin{array}{r} 9 \\ 7 \\ 5 \\ 8 \\ 6 \\ \underline{3} \end{array}$$

Eye movement in column addition. The characteristics of the eye movements are as revealing in addition as they are in reading. Extensive experimentation on eye movement has been conducted by the School of Education of the University of Chicago. Buswell describes the mechanism for recording these movements in addition,

> In brief, the method consists in the photographing of a light from a Tungsarc bulb, which is reflected to the cornea of the eye from silvered-glass mirrors and from the cornea of the eye through a camera lens to the back of the camera where it falls on a continuously moving kinetoscope film. The direction of the pencil of light changes with each movement of the eye. . . . An electrically driven tuning fork, built to vibrate 25 times per second, is interposed in the path of the beam of light in such a way that the pencil of light is intercepted at each vibration. There is thus produced on the film a series of dots, . . . each dot representing a twenty-fifth of a second.[9]

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[9] G. T. Buswell, "Diagnostic Studies in Arithmetic," *Supplementary Educational Monographs*, No. 30 (University of Chicago Press, July, 1926), pp. 14–15.

This record of eye movement is significant. Since the child does not know what is being sought, his eye movements are not conditioned by any disturbing factor. The record of eye movement thus gives an insight into the child's mental behavior in a specific arithmetical situation.

A study of the characteristics of eye movement in adding leads to the following generalizations:

1. In efficient adding there is a regular movement from top to bottom or from bottom to top. We also find that (*a*) the fixations are in sequence; (*b*) there are as many fixations as digits to be added; (*c*) the intervals between fixations are fairly regular and give opportunity for combinations of numbers—thus a 3 and a 2 are seen as 5.
2. In poor performers the movement shows regressions, that is, the fixations are not in the directions desired by the pupil, the number of fixations is greater than the number of digits in the column, the intervals of fixation are irregular and show marked increases at moments of confusion. A considerable number of random movements are also made during addition by those who have not yet acquired the necessary number habits.

Interesting comparisons can be made of eye movements in reading with those in addition. The lesson from the facts thus far acquired is simple: we must impress upon each child the importance of continuing his addition in one sequence, say, from top to bottom, and of resisting any temptation to look back or to linger at any point in the column of figures.

Adding up or down. Whether the eye should move up or down in adding a column of figures is a question which has engaged the attention of many research workers. The dispute seems trivial[10] and the investigations concerning it seem footless. The data gathered so far favor a downward movement of the eye.[11] But the most practical method of checking addition is to add each column in reverse order. Hence children must be taught to add downwards and then upwards. The important characteristics to develop in eye movement in adding, are consistent direction and regularity of movement and fixation.

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[10] F. B. Knight, "Some Aspects in Elementary Arithmetic," *Curriculum Problems in Teaching Mathematics, Second Yearbook* of the National Council of Teachers of Mathematics (Teachers College, Columbia University, 1927), p. 11.

Gradation of single-column addition. With children other than the very ablest, it is well to grade single-column addition by limiting the sums to 20, then to 30, and thus increasing them by decades. Finer gradation than this is unnecessary. Nothing will take the place of regular daily practice in addition.

Two or more column addition. *Gradation.* At first children should be taught to add two or more columns in which there is no carrying. Such examples are:

24		112
31	and	314
32		121
12		42
—		—

A variation of this type of example introduces numbers in the left column which will yield an additional place, thus:

24	412
53	314
72	521
—	42
	—

The next major step is taken when carrying is necessary in one column. The succeeding steps in the gradation are offered earlier in our study of written addition.

Carrying. In teaching children to add the accompanying numbers of two orders, review the names of the two places, units and tens, or cents and dimes. When the sum of the units is obtained let the children analyze 17 into 1 ten and 7 units or 1 dime and 7 cents. The simplest procedure is to put cents or units in the units' column and the ten or dime in the tens' column. The carry figure is written in the appropriate column.

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[11] B. R. Buckingham, "Adding Up or Down," *Journal of Educational Research,* Vol. 12, November, 1925, pp. 251–261; "Arithmetic at the Crossroads," Ohio State University *Educational Research Bulletin,* Vol. 6, January 5, 1927, p. 6.

$$\begin{array}{r} 54 \\ 43 \\ 29 \\ 31 \\ {\scriptstyle 1} \\ \hline 7 \end{array}$$

Objection is made to the use of the auxiliary carry figure on the ground that it is a crutch. In defense we may argue that the object is to attain accuracy and reasonable speed. To insist that children "keep the carry figure in mind" may retard their speed by causing them anxiety lest they forget to carry. A cause of error is thus introduced. Bookkeepers always put their "carry figures" at the foot of the ledger for the help these figures render in checking addition. The critics of the procedure here suggested insist that when the addends are numerous and large, carry figures may be inserted in the proper column, but children should not be given such long addition examples. The question is thus raised whether we should teach two addition practices or techniques, depending on the length of the example, or only one and then guide children to apply it consistently. Surely the example at the right is not beyond the scope of the elementary school. Writing the carry figure in the tens' place is a decided help to any child and enjoys a good reputation among bookkeepers.

$$\begin{array}{r} 54 \\ 73 \\ 89 \\ 97 \\ 21 \\ 83 \\ {\scriptstyle 2} \\ \hline 7 \end{array}$$

A device to quicken the habituation of the "carrying" technique is to give practice, as indicated below, for a few minutes on each day of one week. The following is placed on the board:

4 7 2 5 9 1 8 3 6

Always add $\boxed{3}$ to the answer.

The teacher points to 4, 5, 8, and 6 and does not accept the answer 23, but insists on 26, that is, 23 increased by 3. She then points to 9, 6, and 4; she rejects the answer 19 and accepts only 22 (19 increased by 3). After five or six such examples, the 3 in the carry box is changed to another number and the practice is continued. Three or four minutes daily for five or six days facilitates the learning of carrying in addition.

In teaching "carrying" we usually explain the process with examples in which the carry number is 1. As an initial step, this is thoroughly justified. But at a later time it becomes important to teach children how to carry 2, 3, 5, and all other numbers to 10. In an advanced stage in addition the carry numbers should be larger than 9. Children in the fifth grades obtain 127 as the sum of a first column. They put down the 7 correctly and look puzzled when they find that they have 12 to carry. This condition must be anticipated so that the child will meet it intelligently when it arises.

In teaching three-column addition a similar procedure may be followed. The processes should be rationalized, but the rationalization should be kept simple and the algorism should be designed to aid the acquisition of the necessary habits. It seems unwise to overanalyze and dig too deep in explaining the process. Such rationalization as the following merits the general condemnation now accorded it: 238+325=

$$\begin{array}{rcll}
238 &=& 200 + 30 + 8 & \\
325 &=& 300 + 20 + 5 & \\
\hline
&=& 500 + 50 + 13 = & 500 \\
&&& 50 \\
&&& \underline{13} \\
&&& 563
\end{array}$$

Even the superior children are lost in this maze of equivalents and cannot make the synthesis of the many steps necessary for a generalization.

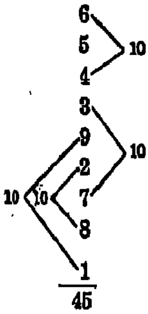
Miscellaneous drills and suggestions. As soon as a sufficient number of habits in additive combinations have been developed, any appropriate form of addition employed in commerce or in industry should be used for systematic practice. Mechanical devices like the ladder, the circles, the triangle, the staircase, and the railroad ties have their values, but they give practice in only one kind of addition. The child must learn to do more than add 9, 6, 5, 8, 7, 4. Hence statistical data found in geography or history texts or in newspapers, and conveniently arranged figures in the arithmetic textbook will be found helpful in providing continued practice in many kinds or arrangements of addition.

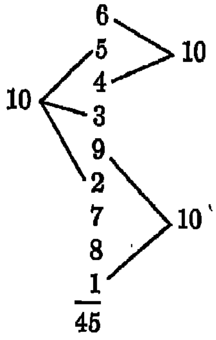
Practice may also be given, though not very frequently, in horizontal additions by requiring pupils to make tabulations like the following:

Record of Parcels Shipped by Brown & Co., Week Ending October 27

Sent by	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total for the Week
Parcel post..	72	148	106	203	117	48	?
Express.....	38	96	92	85	116	71	?
Freight......	17	32	41	54	74	39	?
Messenger...	9	7	14	12	8	27	?
Daily totals	?	?	?	?	?	?	?

Systems of adding have been devised for increasing speed. So far the most reliable aid in achieving reasonable rate is the acquisition of those habits that make for accuracy. For this reason the much extolled "catching ten's" should not be taught. Speed may be attained by this process at the cost of accuracy. Too often the child spends too much time looking for combinations that make ten, and thus loses time. The accompanying examples illustrate the "method of catching ten's." A mere glance shows that





unless the child exercises unusual care, some numbers may be overlooked in gathering the addends that produce a ten.

Practice in rapid addition of two addends of two orders (for example, 35+48) is advocated by many. In such addition two processes may be used: *Form A:* 30+40=70, 5+8=13, 70+13=83; *Form B:* 35, 75 (35+40), 83 (75+8). In *Form A* there is danger that the child will lose the 70 (resulting from 30+40), by the time he has 13 (resulting from 5+8). The second form is recommended because it is a continuous process and the child can be taught to think "35, 75, 83." Addition of three numbers of two orders each can be performed in the same way; namely, 54, 54+20=74, 74+8=82, 82+10=92,

$$\begin{array}{r}54\\28\\\underline{19}\end{array}$$

92+9=101; the child thinks: "54, 74, 82, 92, 101."

This form of rapid addition must be graded before it is taught. The following tabular arrangement summarizes a plan of gradation discussed earlier.

Step I	Step II	Step III	Step IV
30 + 40 60 + 20	34 + 40 57 + 20	34 + 43 57 + 22	34 + 47 57 + 28
Basis: 3 + 4 and 6 + 2	Basis: Step I	Units numbers give sum not greater than 9.	The ten's are bridged.
Child thinks: 30, 70; 60, 80	Child thinks: 34, 74; 57, 77	Child thinks: 34, 74, 77 (34 + 40 + 3); 57, 77, 79 (57 + 20 + 2)	Child thinks: 34, 74, 81 (34 + 40 + 7) 57, 77, 85 (57 + 20 + 8)

Five minutes out of each arithmetic lesson for a week, devoted to this graded practice, develops ability to add rapidly and continuously such numbers as 28 and 46, 37 and 59, and 44 and 37.

Inventory of common errors in addition and remedial suggestions. In every arithmetical process, there are common pitfalls. These we must know in order to forestall them in our teaching and to look for them in the work of less proficient pupils. We shall present these common errors and suggest corrective measures for each.

The Common Errors	*The Corrective Measures*
1. Errors due to faulty learning of the basic combinations. $7 + 8 = 16$ (ignorance of number facts or of the signs) $5 + 1 = 5$ (confusion with multiplication) $\begin{array}{r}59\\ 40\\ \hline 90\end{array}$ (zero difficulty) $\begin{array}{r}47\\ 39\\ 67\\ \hline 152\end{array}$ (imperfect mastery of carrying basic combinations to higher decades)	1. *a.* Give periodic diagnostic tests on basic combinations to discover individual weaknesses. *b.* Encourage pupils to keep an individual record of combinations requiring further drill. Supply differentiated drills. *c.* Give special periodic drills to the class on the most troublesome combinations as revealed in diagnostic tests. *d.* Discourage counting and such auxiliary aids as $9 + 7 = 16$ because $9 + 1 = 10$, 1 taken from 7 leaves 6, hence $10 + 6 = 16$. Some children indulge in even more complicated arithmetical circumlocutions. *e.* Periodic tests to ascertain degree of mastery of the symbols +, =, etc.
2. Errors due to defects in carrying. $\begin{array}{r}86\\ 95\\ \hline 171\end{array}$ (failure to carry) $\begin{array}{r}35\\ 23\\ 68\end{array}$ (carrying when not necessary) $\begin{array}{r}83\\ 49\\ \hline 141\end{array}$ (carrying the wrong number, $9 + 3 = 12$. The 2 was carried.)	2. *a.* Give a simple explanation of carrying, and provide drills in carrying as suggested earlier in this chapter. *b.* Vary types of examples so that in any one lesson there are examples that do not require carrying as well as examples that do. *c.* Insist on checking of answers. *d.* Much individualized and differentiated drill on upper-decade addition.

The Common Errors	*The Corrective Measures*
3. Errors due to carelessness resulting in misreading of directions and in miscopying. $7 + 8 = 56$ (+ read as ×) $6 \times 4 = 10$ (confusion due either to lack of knowledge of symbols or to careless reading of symbol) Add: 78 8 624 (multiplication performed) Add 74, 36, and 42. 74 36 24 (42 is miscopied)	3. *a.* Occasional talks on the importance of careful reading of examples, figures, symbols, and key words are helpful. *b.* Give frequent examples in addition with only two addends so that children will not associate any two numbers with multiplication. *c.* Once each week a five-minute exercise should be assigned in copying such numbers as 339, 933, 903, 309, 711, 117, 171.
4. Errors due to skipping or repeating decades. $37 + 8 = 55$ (skipping the forty decade) $37 + 8 = 35$ (repeating the thirty decade)	4. *a.* Insist on checking of every answer by doing example again or by adding in reverse order. The errors referred to in the left column do not occur in isolated form as there shown, but in the process of adding long columns. *b.* Motivated practice in carrying basic combinations to higher orders is a decided help to children who make these errors.

These are not the only errors made by children, but they are the most common errors. A complete list of possible errors in addition reaches a total of over forty-four.[12] Such a list is interesting but excessively full for practical classroom use. One cannot and need not teach every component skill that enters into the general ability known as addition. We teach each of the important skills and assume, with justification, that a measure of transference does occur.[18] When a child has learned to carry 1, 2, 3, and 6 we may rest assured that he will know what to do in an example in which 4 or 5 must be carried. We shall refer to these common errors later in our discussion of diagnostic testing and remedial teaching.

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[12] B. F. Knight, "Some Considerations of Method," in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education (Public School Publishing Co., 1930), pp. 177–179.

PROBLEMS FOR STUDY AND DISCUSSION

1. Three positions may be taken with reference to transfer of training in number combinations:
   (*a*) No transfer takes place: the child who learns 6 + 7, is not helped in learning 7 + 6 or 17 + 6 or 16 + 7.
   (*b*) Training is general and therefore transfer of training takes place: the child who learns 6 + 7 will know 7 + 6 and 17 + 6 and 16 + 7 if he is attentive while learning.
   (*c*) A measure of transfer of skill does take place because of the similarity in number situations: the child who learns 6 + 7 may not know 7 + 6, but by the very learning of the fact 6 + 7, he facilitates the learning of 7 + 6 and even 16 + 7.

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[18] An interesting report of an experiment made with 100 children of 2B grade is presented by Beito and Brueckner, "A Measurement of Transfer in the Learning of Number Combinations," *Twenty-Ninth Yearbook* of the National Society for the Study of Education, Part II, Ch. vii. The investigators come to the following conclusions: "(1) When pupils of any mental level are taught only the direct form of an addition combination such as $\begin{array}{r}7\\ \underline{4}\end{array}$ as nearly as can be, the reverse form, $\begin{array}{r}4\\ \underline{7}\end{array}$ is learned concomitantly at least as completely as the direct form. (2) The bond formed in learning the direct form of an addition combination carries over almost completely to the reverse form. The amount of carry-over is influenced very little by the method of presentation." We believe that the amount of transfer is influenced by the degree to which one generalizes about identical elements. In this investigation the greatest transfer occurred in the inferior group. The limited scope of the experiment makes these findings inconclusive.

Shorling, Overman, and Shriner (editors), *Contemporary Guide in the Teaching of Arithmetic*, Units 12 and 13 (Edwards Bros. Inc., 1933). These two units are extremely helpful, especially the chapter on "Transfer of Training in Arithmetic" by Overman.

Which of these three positions are we warranted in accepting on the basis of experimental data thus far gathered? Give reasons and references.

2. Formulate a justification for the following procedures:

(*a*) Teach $7 + 9$ and $9 + 7$ in the same lesson, as related facts rather than in separated lessons as discrete facts.

(*b*) To teach $5 + 8$, begin with a single large group of 13 objects and not with two small groups of 5 and 8 objects.

(*c*) Zero combinations in addition may well be postponed until the child does column addition.

(*d*) Be cautious in providing practice in addition through games.

(*e*) Discourage voiced or unvoiced speech in addition, even though the pupil may insist that he can add more rapidly and more accurately that way.

3. Make a list of the technical terms and expressions which occur in this chapter. Define and illustrate each.

4. List the characteristics of eye movement in effective reading; compare these with the eye movements of a pupil who is efficient in adding. Repeat the comparison, using the eye movements of children deficient in reading and addition.

5. Secure a set of test papers in column addition. Analyze the errors you find in children's work, following the analysis of errors offered in this chapter. For each common error, set forth corrective measures.

6. If possible, have assigned to you a small group of about five children who are poor in addition. Preparo a diagnostic test in addition. Obtain the I.Q. or the mental age of each child. What diagnosis can you make of each child's deficiency? What corrective program do you recommend?

CHAPTER XXI

TEACHING SUBTRACTION OF WHOLE NUMBERS

Oral Subtraction

Relation of subtraction to addition. While there are no objective data on which to decide whether addition and subtraction combinations should be taught separately or together,[1] experience seems to indicate that the latter procedure is the wiser. When children have mastered the 100 basic combinations of addition, they have taken a significant step towards the mastery of the subtraction combinations. Hence $9+6$ and $6+9$ are not really learned until the children know $9+?=15$ and $?+6=15$. Another step brings the pupils to 15 less 6 and 15 less 9.

The basic combinations of subtraction. In the learning process the foundational facts of subtraction come, as we have just seen, from the basic combinations of addition. A careful review of these additive combinations, with stress on the forms for subtraction, is sufficient at this stage. Care must be exercised to teach each form of expressing the idea of subtraction. Thus, to teach $16-7$, we follow the lesson outlined for any combination of addition (see pages 256–262) and then introduce the following at the appropriate place:

7 cents and how many more make 16 cents?	$7+?=16$
How many cents and 9 cents make 16 cents?	$?+9=16$
I had 16 cents and lost 7 cents.	$16-7=?$
I had 16 cents and lost 9 cents.	$16-9=?$

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[1] B. R. Buckingham, "Teaching Addition and Subtraction Facts Together or Separately," Ohio State University *Educational Research Bulletin*, Vol. 6, May 25, 1927, pp. 228–229, 240–242.

I had 16 cents yesterday and only 7 cents to-day; how much did I lose? $16 - ? = 7$

I had 16 cents yesterday and only 9 cents to-day; how much did I give away? $16 - ? = 9$

Both the vertical and the horizontal or equation forms of the algorism should be taught, that is,

$$\begin{array}{r} 16 \\ -7 \\ \hline ? \end{array} \quad \begin{array}{r} 16 \\ -9 \\ \hline ? \end{array} \quad \begin{array}{r} 16 \\ -? \\ \hline 7 \end{array} \quad \begin{array}{r} 16 \\ -? \\ \hline 9 \end{array} \quad 16-7=?,\ 16-?=7, \quad \begin{array}{r} ? \\ +9 \\ \hline 16. \end{array}$$

Finally, the comparative form of subtraction must be introduced: 16 is how much greater than 9? 9 is how much less than 16? Through a host of practical examples that lend themselves to objective demonstration the child should be made familiar with the various forms of expressing the idea of subtraction.

Gradation of the basic combinations. The gradation of the 100 combinations in addition applies, of necessity, to the subtraction combination. One change is urged, to wit, that all zero combinations be taught last and in the following sequence: (*a*) a number less itself, $\begin{array}{r} 5 \\ -5 \\ \hline 0 \end{array}$ and $5-5=0$; (*b*) a number diminished by 0, $\begin{array}{r} 5 \\ -0 \\ \hline 5 \end{array}$ and $5-0=5$; (*c*) zero less zero, $0-0=0$. In all other respects the gradation of addition may be followed.

The three basic ideas of subtraction. From the illustrations we have cited, three distinct subtraction ideas emerge:

1. What is left—I had 12 and lost 3; how many are left?
2. How much is needed—I have 12 but I need 15; how many more must I get?
3. The difference—I have 12 and John has 15; how many more has John?

These three conceptions of subtraction must be repeated in simple, practical problems as children learn each new combination. Unless this is done children will know the abstract number facts but will be helpless to make applications of them to real situations.

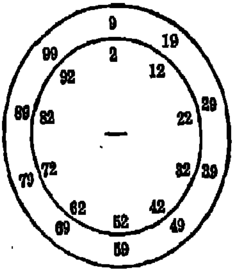
Widening the scope of the basic combinations. Unless the foundational combinations are carried to higher decades their usefulness is appreciably lessened. Many who can give an instantaneous response to 17−8 hesitate at 37−8 and say "20, 20, 29." There is an explanation but not an adequate excuse for this retarded response to 37−8—failure to carry basic combinations to higher decades.

It is well to grade these subtraction combinations of higher decades. The following plan is suggested:

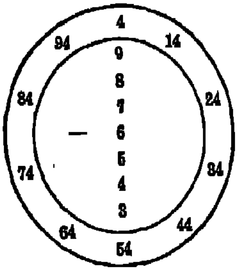
Step I. Combinations in the same decade: 59−52; 67−61; 86−84.
Step II. Subtrahend, 10, 20, 30, 40, etc.: 78−10; 78−20; 78−40.
Step III. Combinations with a zero in the minuend: 60−52; 70−61; 90−84.
Step IV. Bridging ten's, with subtrahend over ten: 52−11; 58−12; 74−13; no borrowing involved.
Step V. Bridging ten's with a subtrahend less than ten: 62−9; 54−8; 83−7.
Step VI. A combination of steps IV and V: (*a*) 74−68; 82−76; (*b*) 74−16; 82−15.

The gradation may be made finer or cruder depending upon the ability of the pupils. The elimination of Steps IV and VI can be justified very readily.

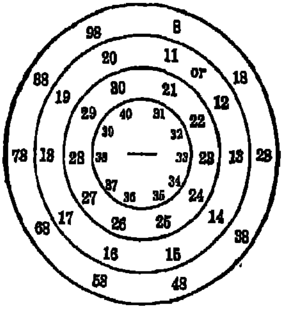
Mechanical devices will facilitate the drill on higher-decade subtraction. A few illustrations are here submitted:



Subtraction in the same decade. 9−12, 19−12, 69−62, 89−82, 99−92.



Subtraction of a digit from any decade. 9 from 14, 24, 64, 54, 94. 5 from 14, 94, 84, 74.



Subtraction of any number of two orders from any other number of two orders. 38−19, 68−56, etc. Process: 38−19, 38−10, 28−9, 19. Reverse of similar addition device.

The numbers on the large dial must be changed in each case until every variety of combination is introduced.

The following is a simple but helpful preparatory device:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & & & & & & & & -8 \\ \hline \end{array}$$

The teacher points to 9 and −8 or to 8 and −8, child gives 1 or 0. Teacher points to 7 and −8; when child says, "can't take 8 from 7" the teacher gives permission to make any number ten higher, for example, 6 becomes 16, 7 becomes 17, 4 becomes 14. Teacher points to 3 and −8, to 5 and −8, to 4 and −8, to 6 and −8, and the child thinks "13, 15, 14, 16" as the minuend in each case, and gives the answers, "5, 7, 6, 8" respectively. The subtrahend 8 is changed to another number and the drill is repeated until the responses become habitual.

This is a useful drill to prepare for subtraction with borrowing:

$$\begin{array}{r} 52 \\ -27 \\ \hline \end{array} \qquad \begin{array}{r} 52 \\ -\,? \\ \hline 27 \end{array} \qquad \begin{array}{r} ? \\ -27 \\ \hline 25 \end{array} \qquad \begin{array}{r} 27 \\ +\,? \\ \hline 52 \end{array} \qquad \begin{array}{r} ? \\ +27 \\ \hline 52 \end{array}$$

52 is how much greater than 27? 27 is how much less than 52? Required process: (*a*) 52−27; 52−20, then take 7 from the result; for example, child thinks "52, 32, 25." (*b*) Addition idea—what number added to 7 gives two, etc., as will be explained under Austrian subtraction.

Present	*Obtain*		
	a	*b*	*c*
52−27	52−	27=	25
92−60	92−	60=	32
74−26	74−	26=	48
	218−	113=	105

Call first column *a*, second *b*, the answers *c*; then add all of *a*, all of *b*, all of *c*; take all of *b* from all of *a* to find if total *c* results.

218−113, 218−100=118
118− 10=108
108− 3=105

To increase the number of these drills special adaptations can be made of the devices suggested in addition. These drill forms are for all grades throughout the school course.

Written Subtraction

Gradation.

1. No carrying

Gradual increase in number of places with zeroes in the subtrahend: $\begin{array}{r}89\\ \underline{72}\end{array}$ $\begin{array}{r}894\\ \underline{273}\end{array}$ $\begin{array}{r}546\\ \underline{203}\end{array}$, etc. There is no difficulty of number or of teaching in this type of subtraction.

2. Carrying or borrowing

a. Zero in units' place, $\begin{array}{r}70\\ \underline{35}\end{array}$ $\begin{array}{r}80\\ \underline{43}\end{array}$

b. Number larger than zero in unit's place in two-place numbers, $\begin{array}{r}71\\ \underline{35}\end{array}$ $\begin{array}{r}82\\ \underline{43}\end{array}$

c. Three-place numbers with borrowing in the tens' place, $\begin{array}{r}837\\ \underline{245}\end{array}$

d. Steps 2 and 3 combined, $\begin{array}{r}836\\ \underline{247}\end{array}$ $\begin{array}{r}754\\ \underline{267}\end{array}$ also $\begin{array}{r}754\\ \underline{206}\end{array}$

e. Three-place numbers, internal zero in minuend, $\begin{array}{r}806\\ \underline{244}\end{array}$

f. Steps 2, 3, and 5 combined, $\begin{array}{r}806\\ \underline{247}\end{array}$ $\begin{array}{r}605\\ \underline{458}\end{array}$

g. Increase places with one or more difficulties in the list above.
h. Two or more zeros either terminal or interval,

$\begin{array}{r}800\\ \underline{274}\end{array}$ $\begin{array}{r}8000\\ \underline{2746}\end{array}$ $\begin{array}{r}8004\\ \underline{3876}\end{array}$ etc.

i. *Any reasonable subtraction example.*

Teaching of subtraction with borrowing or carrying. There is obviously no difficulty in the first type of subtraction when every number in the subtrahend is smaller than the number in the corresponding place in the minuend. Here the child writes the answer instead of saying it. The process is familiar; only the algorism must be taught.

But when a number in the subtrahend is larger than the number in the corresponding place in the minuend, difficulty is experienced by the children, and a teaching problem arises.

It is best to begin this type of subtraction with minuends having zero in the units' place, $\begin{array}{r}70\\ \underline{28}\end{array}$, $\begin{array}{r}60\\ \underline{34}\end{array}$, because only one major new step must be taught. In such examples as $\begin{array}{r}72\\ \underline{28}\end{array}$ two new steps are involved.

There are many ways of subtracting 28 from 72. Which to teach is still a moot question. The reader is advised to perform each of the following examples slowly so that he may identify the process he uses.

1. *The method of take away and carry or equal additions.*

$\begin{array}{r}72\\ \underline{-58}\end{array}$

The rationalization: 8 from 2, I can't take. I borrow 1 from the tens' place, which gives me 10 units. 2 units and 10 units are 12 units. 8 from 12 is 4. But the one 10 borrowed must be repaid to the tens' place. Hence, 6 from 7 is 1.

The child thinks: 8 from 2; 8 from 12, 4; 6 from 7, 1.

Criticism: The process is based on the axiom that if equals are added to each of two numbers, the difference between them remains unchanged. In this method we proceed as if we had done the following:

$$\begin{array}{rlrl} \overset{10}{72}, & \text{thus resulting in} & & 70+12 \\ -\overset{1}{5}8 & & \text{minus} & 60+\ 8 \\ \cline{3-4} & & \text{result} & 10+\ 4=14. \end{array}$$

This rationalization does not rationalize the process to young children, because they do not understand why we pay back to the lower 10's after having borrowed from the upper 10's. The conception of borrow and pay back is not mathematical and probably tends to interfere with rapid habituation. In such examples as 3203−1654 this process becomes especially cumbersome.

2. *Method of decomposition or reduction to equivalents.* *The rationalization:* 8 from 2 I can't take, hence change one of the 10's to 10 units. 2 units and 10 units make 12 units. 8 units from 12 units leaves 4 units. 5 ten's from 6 ten's (not 7 ten's since 1 ten was converted into units) leaves 1 ten.

Child thinks: 8 from 2; 8 from 12, 4; 5 from 6, 1.

Criticism: This method is the favorite of many teachers who hold that it is very simple because it can be demonstrated objectively with dimes and cents and that it can be habituated quickly. Others are opposed to it because it requires mastery of a second set of number facts—the subtraction combinations. They also point out that when two zeros occur in the minuend the children experience no little difficulty in reducing to equivalents, thus:

$$\begin{array}{r} \overset{2}{\cancel{3}}\overset{9}{\cancel{0}}\overset{9}{\cancel{0}}\overset{10}{4} \\ -1768 \\ \hline 1236 \end{array}$$

The recommended form

$$\begin{array}{r} 3004 \\ 1768 \\ \hline 1236 \end{array}$$

no "crutches"
should be permitted.

Of course, the auxiliary figures are here shown merely to indicate the child's thinking as he carries out each step. Surely no one will maintain that this is a simple process. These two indictments are swept aside by the proponents of the method of decomposition or reduction-to-equivalents, who argue: subtraction tables are necessary in all methods, and a minuend like 3004 usually presents difficulties no matter what system of subtraction is used.

3. *Subtraction by addition, the Austrian method.* Earlier in our discussion subtraction was presented as a process of comparison as well as a process of diminution. A fuller meaning of these two concepts may be obtained by tracing their contrasting characteristics:

A	*B*
John had 25¢ and lost 20¢. What had he left?	John had 25¢, William 20¢. John has how much more than William?
1. The subtrahend was originally part of the minuend.	1. The subtrahend is a separate quantity and was never part of the minuend.
2. The result, 5¢, is a remainder.	2. The result here is a measure of difference.
3. This example requires 25¢ for objective presentation.	3. This example requires 45¢ for objective presentation.
4. The process, 25¢ — 20¢ = 5¢, is one of diminution.	4. The process, 20¢ + ? = 25¢, is one of comparison and difference.

It is obvious that subtraction by adding or by comparing must be stressed. The Austrian method holds that subtraction can be reduced to addition, and therefore should be taught, not as a new process, but as an application of the older one. "John has 5c and William 7c, how much does John need to have as much as William?" reduces itself, naturally, to John's money plus how much will equal William's, or 5+?=7. If the child knows the 100 combinations of addition, he knows the missing number, 2. Let us analyze a few examples worked by additive subtraction.

The rationalization: 2 and what will give 6? 4. Write 4 in answer. 4 and what gives 7? 3. Write 3 in answer. 1 and what will give 3? 2. Write 2 in answer.

The child thinks: 2 and 4, 6; 4 and 3, 7; 1 and 2, 3.

376
142
234

The rationalization: 8 and what will give 6? Nothing. 8 and what will give a 6? 8. 8 and 8 are 16, hence write 8 in answer and carry 1 to the tens' column. 1 and 2 are 3. 3 and what will give a 4? 1. Write 1 in answer. 1 and what will give 3? 2. Write 2 in the answer.

346
128
1
218

Child thinks: 8 and 8, 16; 2, 3, 3 and 1, 4; 1 and 2, 3.

The rationalization: 8 and what gives 4? 8 and what gives a 4 or 14? Answer 6. Write 6 in answer. 6 and 8 are 14, carry 1 to tens' place. 6 and 1 are 7. 7 and what gives 2? a 2 or 12? answer 5. Write 5 in answer. 5 and 7 (6+1 carried) are 12, hence, as in addition, carry 1 to hundreds' place. 4 and 1, 5. 5 and what gives 7? 2. Write 2 in answer.

724
468
1 1
256

Child thinks: 8 and 6, 14; 6, 7, 7 and 5, 12; 4, 5, 5 and 2, 7.

Critical analysis of the Austrian or additive method. The superiority of any one of these methods of subtraction has not as yet been proved. Opinion is still pitted against opinion and the debate continues.

The case for the additive method. 1. No new number combinations need be taught. The child learns 6+2=8 in addition and uses this combination in 8−2 or 8−6 because he thinks 2 and ? gives 8 or 6 and ? gives 8.

2. It is the method that experience teaches the small shopkeeper in making change. When the purchaser of a 69-cent article gives the merchant a dollar, the latter gives the former the amount necessary to add to 69 cents to equal a dollar, by counting "69 and 1 makes 70, and 5 makes 75 and 25 makes 1 dollar." In getting his ledger balance he always seeks the number which, added to the smaller, will give the larger sum.

3. Additive subtraction proves the answer in the very

process of subtraction. In other methods of subtraction the proof consists of adding the answer and the subtrahend to obtain the minuend. In the Austrian subtraction this addition goes on all the time so that when the subtraction is complete every item has been automatically checked.

4. Investigators—not all, to be sure—have found additive subtraction superior to other methods. Their testimony, concerned with *equal additions*, is interesting. "For every age the E. A. [equal addition] children of both sexes were found to work subtraction more expeditiously than the D [decomposition] children. And on the whole the number of errors is less."[2] McClelland concludes: "The method of equal addition appears superior in speed, accuracy, and adaptability to new conditions, while the method of decomposition is superior in speed after long practice."[3] Winch tells us that the method of equal addition is superior in accuracy and rate with young children. "This is true in the case of the superior children who already had learned something of both methods, and also in the case of the inferior children, who, prior to the experiment, really knew nothing of either method."[4] The proponents of the Austrian method ought to derive little comfort from these results. They apply, strictly speaking, to the method of equal addition, and were derived from experiments with small groups of children.

The case against additive subtraction. 1. Osburn[5] and

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[2] P. B. Ballard, "Norms of Performance in the Fundamental Processes of Arithmetic, with Suggestions for Their Improvement," *Journal of Experimental Pedagogy,* Vol. 2, December, 1914, pp. 396–405; Vol. 3, March, 1915, pp. 9–20.

[3] William W. McClelland, "An Experimental Study of the Different Methods of Subtraction," *Journal of Experimental Pedagogy,* Vol. 4, December, 1918, p. 298.

[4] W. H. Winch, "Equal Additions vs. Decomposition in Teaching Subtraction," *Journal of Experimental Pedagogy,* Vol. 5, December, 1920, p. 270.

[5] W. J. Osburn, "How Shall We Subtract," *Journal of Educational Research,* Vol. 16, November, 1927, pp. 237–246.

Buckingham,[6] on the basis of their own studies, conclude that the "take-away" method is *more* effective than the additive. But, again, their studies are not conclusive for reasons similar to those that prevent us from accepting as final the findings of their opponents. We have, then, equally inconclusive experimental data for both sides of the dispute.

2. The argument that no new number combinations need be taught for additive subtraction cannot be maintained, for it holds with equal effectiveness for "take-away" subtraction. $7+4=11$ is not completely mastered unless we also teach $7+?=11$ and $?+4=11$. The reverse of any number combination, we saw, is learned most economically when taught with the direct form. In neither system of subtraction can the idea involved in $11-4=?$ be neglected, because in dealing with quantities the act of diminishing is as basic as the act of augmenting. In both systems every additive combination, for example, $7+4=11$, must be modified to $7+?=11$, then to $11-7=?$ and finally to $11-?=4$.

3. Everyday business transactions do not demand additive subtraction. True, the shopkeeper does not usually say to his customer who makes a purchase amounting to 69 cents and presents a dollar, "$100-69=31$." But neither does he say "$69+?=100$." He uses coins and builds up from 69 to 100 by adding 1 cent and then either 3 dimes or 1 quarter and 1 nickel. Surely this process of adding the coins is not additive subtraction in the sense in which this term is employed. In business practice subtraction is usually regarded as diminution. This is true in all forms of discount, depreciation of values, business losses, depleted stock, and the like. In these situations one generally thinks of a larger amount that became less.

4. To hold that in the additive method one proves his work

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[6] B. R. Buckingham, "The Additive vs. the Take-away Method of Teaching Subtraction," Ohio State University *Educational Research Bulletin*, Vol. 6, September 28, 1927, pp. 265–269; "Arithmetic at the Crossroads," Ohio State University *Educational Research Bulletin*, Vol. 6, January 5, 1927, p. 6.

as he performs it is to court deception. Let us compare what happens in each of these two methods of subtraction.

Decomposition (Take Away)	*Additive or Austrian*
7203 Minuend (M)	7203
−4627 Subtrahend (S)	4627
	111
2576 Difference (D)	2576
7203 Proof, D + S = M	7 and 6 = 13; carry 1 to 2, result 3; 3 and 7 = 10; carry 1 to 6, result 7; 7 and 5 = 12; carry 1 to 4, result 5; 5 and 2 = 7.

At first glance it does seem as if the very process of work proves the subtraction in the Austrian method, but a moment's thought reveals that the inference is unjustifiable. What if the child says: 7 and ? = 13; answer, 9; 9 and 7 = 13, write 9 and carry 1. When an error is made in Austrian subtraction it is affirmed by repetition of the same combination. In the "take-away" subtraction the proof is a genuine check because the child uses two processes, 7 from 13 = ? and 7 and 6 = ?

5. The additive system of subtraction is not carried out consistently. In long division and in subtraction of fractions many children drop additive subtraction for a "make less" system. Taylor found that by the time "children reach the sixth grade, 88 out of every 100 taught by the Austrian method subtract by a method (equal additions or decomposition) not taught them."[7]

Summary of the dispute. It is clear that there are two prevailing methods of teaching subtraction: the "take-away" and the additive or Austrian. Whether one is significantly superior to the other is an open question that no debate will settle. Nor will speculations on the origin of number ideas help. We are told that subtraction originated in counting backwards. The statement is accepted as fact in some quarters and inferences are drawn from it. Who knows how the subtraction idea arose? An apparently logical supposition is no assurance of actual occurrence. What if subtraction did originate in counting backwards? Are we to teach it that way? Shall we follow the culture-epochs theory to such a degree? The argument runs: the "take-away" process should be taught first "because of the psycho-genetic" implication. All this is a *non sequitur*. The dispute will be settled only by a well planned experimental study with suitable controls. Until we have such data each contestant must rely on the force of verbalism rather than on fact. It may be that we shall find a modified plan necessary in which the vital point will be to teach children the "take-away" process, but to increase the subtrahend figure by 1 rather than to diminish the minuend figure by 1, thus,

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[7] J. S. Taylor, "Subtraction by the Additive Process," *Elementary School Journal*, Vol. 20, February, 1919, pp. 203–207.

$$\text{Think 6 from 8 not 5 from 7} \longrightarrow \begin{array}{r} 849 \\ \underline{556} \\ 93 \end{array}$$

Whatever the conclusion, it must be derived by a scientific method.

General suggestions for teaching addition and subtraction. Throughout the work in addition and subtraction proof and checks should be required of the pupils. The relative worth of various methods of proving and checking mechanical operations will be treated at the end of the discussion of the four fundamental processes.

The aim throughout this early work should be to mechanize the process as soon as possible. Rationalizations must always be kept extremely simple and given only to those children who can comprehend them.

In the attempt to habituate a process, teachers must not lose sight of the practical application. Children must not only be able to subtract 289 from 508 with accuracy and reasonable speed, but must also recognize the subtraction idea in practical situations. Too many children become experts in 17−9 and in subtracting 468 from 703 but are helpless when the fact or the process must be applied. Abundant practice in solving problems that involve real situations is very important.

A final caution is to beware of emphasizing difficult forms at the expense of simpler and more common number situations. Children who can divide 54.346 by 2.007 are often confused when asked to divide .1 by .01 or 2 by .2. So, too, children who can subtract 2465 from 5009 err when they are told to take 3564 from 8689, because they continue to "borrow and carry" from force of habit. The business needs of arithmetic abound in simple rather than complicated manipulations.

Inventory of common errors in subtraction and remedial suggestions. Detailed analyses of the general skill known as "ability to subtract" have been made. We reproduce below one of these studies in which subtraction ability is resolved into thirty-two unit skills. To be able to subtract correctly and with reasonable speed a pupil must master each of these thirty-two skills. The list is given here for various reasons: (1) It illustrates the type of analysis that must be made for each fundamental operation. (2) It points to the soft spots in subtraction which need most drill. (3) It reveals the major steps that must be considered in grading subtraction.

Theoretically, children ought to be taught each unit skill in developing the hierarchy of skills that constitutes subtraction. Practically no such fine gradation is necessary, for there is sufficient similarity among many of these progressive steps to insure transfer of a skill in one constituent step to a higher step.

Analysis of Subtraction of Whole Numbers in Terms of Learning Process[8]

The Constituent Skill	*Unit of Skill Number*
I. As to form of stating example	
A. Number written with figures	
1. Indicated subtraction, as $6-2=4$	1
Additive subtraction, as $6-?=4$	1a

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[8] F. B. Knight, "Curricula Problems and Their Scientific Solution," and "According to What Criteria Should Drill Be Organized," in the *Third Yearbook* of the Department of Superintendence, pp. 66–69.

The Constituent Skill	*Unit of Skill Number*
2. Column subtraction, as $\begin{array}{r}6\\2\\\hline 4\end{array}$	2
$\begin{array}{r}6\\?\\\hline 4\end{array}$	2*a*
3. Words used, "minus" or "less" or "take from," as 9 minus 4	3
B. Numbers written with words	
1. Indicated subtraction, as four — two	4
2. Words used, as four "minus," "less," "take from"	5
C. Pictures used	
1. Indicated subtraction, as 0 0 0 — 0 0	6
2. Words used as 0 0 0 — 0	7
II. As to procedure	
A. No borrowing or carrying	
1. One digit, as number — 1-digit number. 6 — 4 =	8
2. Two-digit number — 1-digit number	
When difference is 1 digit, as $\begin{array}{r}12\\-4\\\hline\end{array}$	9
When difference is 2 digits, as $\begin{array}{r}18\\-6\\\hline\end{array}$	10
3. Two-digit numbers less two digits as $\begin{array}{r}48\\-24\\\hline\end{array}$	11
4. When the remainder is zero, as $\begin{array}{r}6\\-6\\\hline\end{array}$ $\begin{array}{r}16\\-6\\\hline\end{array}$	12
5. More than two-digit number, as $\begin{array}{r}483\\-121\\\hline\end{array}$	13
6. Zero in subtrahend, as $\begin{array}{r}463\\-102\\\hline\end{array}$ $\begin{array}{r}463\\-120\\\hline\end{array}$	14
7. Zero in both subtrahend and minuend, as $\begin{array}{r}40\\-20\\\hline\end{array}$	15
8. Gaps in column, as $\begin{array}{r}483\\-21\\\hline\end{array}$	16

The Constituent Skill	Unit of Skill Number
B. Borrowing or carrying	
1. Two-digit number — one digit, as $\begin{array}{r}46\\-6\\\hline\end{array}$	17
2. Two-digit number — one digit Zero in minuend, as $\begin{array}{r}40\\-6\\\hline\end{array}$	18
3. More than two-digit number	
a. Borrowing units column, as $\begin{array}{r}423\\-117\\\hline\end{array}$	19
b. Borrowing other than units column, as $\begin{array}{r}463\\-178\\\hline\end{array}$	20
c. Borrowing two consecutive columns, as $\begin{array}{r}482\\-197\\\hline\end{array}$	21
d. Borrowing two not consecutive columns, as $\begin{array}{r}4236\\-1718\\\hline\end{array}$	22
4. Borrowing zero in minuend, as $\begin{array}{r}420\\-128\\\hline\end{array}$	23
5. Borrowing two zeros in minuend, as $\begin{array}{r}4200\\-1267\\\hline\end{array}$	24
6. Borrowing more than 2 zeros, as $\begin{array}{r}43000\\-12675\\\hline\end{array}$	25
7. Borrowing zero in both not final, as $\begin{array}{r}4306\\-1204\\\hline\end{array}$	26
8. Borrowing zero final, as $\begin{array}{r}248000\\-162000\\\hline\end{array}$	27
9. Subtraction of unequal number digits, as	
a. Where zero is subtracted from last left number, as $\begin{array}{r}4862\\-732\\\hline\end{array}$	28

The Constituent Skill	*Unit of Skill Number*
b. Where zero is not subtracted, as $\begin{array}{r}1467\\-835\\\hline\end{array}$	29
c. Last left numbers are equal, as $\begin{array}{r}635\\-604\\\hline\end{array}$	30
C. Ability to check	31
D. Ability to copy for work as 43728 — 39162 = $\begin{array}{r}43728\\-39162\\\hline\end{array}$	32

For diagnostic testing and remedial teaching of subtraction the reader is referred to an identical topic under addition. For each error listed (page 270) there is a parallel related type of error in subtraction. The reader is urged to work these out and then adapt the suggested corrective procedure to subtraction.

The most frequent causes of errors in subtraction are six:

1. Ignorance of basic number combinations in subtraction.
2. Confusion arising from zeros in these combinations, 5 — 0 = 0, and the like.
3. Inability to apply the basic combinations to higher decades. The child may know 17 — 8 = 9, but 37 — 8 is frequently for him, 9 or 27.
4. Incorrect borrowing 82 — 35 = $\begin{array}{r}82\\-35\\\hline 57\end{array}$
5. Incorrect borrowing when two successive borrowings occur, 824 — 267 becomes $\begin{array}{r}824\\-267\\\hline 567\end{array}$ or $\begin{array}{r}824\\-267\\\hline 657\end{array}$
6. Zeros in minuend and subtrahend $\begin{array}{r}874\\206\\\hline 608\end{array}$ $\begin{array}{r}804\\216\\\hline ??8\end{array}$ $\begin{array}{r}8004\\2406\\\hline ???8\end{array}$

To these we may add ignorance of any of the thirty-two unit skills listed above. These six are major causes that suggest, very readily, the appropriate remedial measures.

Problems for Study and Discussion

1. State your position with supporting reasons on each of the following:

(*a*) It is wise to teach one method of subtraction to make the process clear, and later to teach another for developing speed in computation.

(*b*) It is advisable, in the absence of determining data, to permit each teacher to teach that process of subtraction which she uses in her own computation because we teach best what we know best.

(*c*) In the final analysis, both the "take-away" and the "additive" method should be taught because both are necessary.

(*d*) It is unwise to use money as the only illustrative instrument, useful as the American monetary system may be in teaching.

2. List three important questions in the teaching of subtraction, the solution of which is dependent on experimentally derived data.

3. Plan an experimental inquiry into the relative merits of any two methods of subtraction. Assume that the facilities necessary will be assured. Describe in most precise form, (*a*) the object and the scope of the inquiry; (*b*) the procedure; (*c*) the establishment of "controls" and the attainment of comparable data; (*d*) the cautions to be exercised; (*e*) the type of inference you think may reasonably be drawn from the data; (*f*) the method of checking the conclusions.

4. Consult at least two references to experimental studies in subtraction. Did the authors really prove anything? What? Characterize the method of procedure used by each.

5. Group the 32 unit skills of subtraction here presented in a way which would integrate the list and thus reduce the number of unit skills. Classify the unit skills into, (*a*) most important, (*b*) important, (*c*) relatively unimportant.

6. Analyze the following examples. Identify the elements which will probably cause errors in the work of children:

$$\begin{array}{rrrl} 5824 & 5824 & 5004 & \\ \underline{1209} & \underline{1009} & \underline{1897} & 3042-1979. \end{array}$$

How would you help the pupils to overcome these difficulties?

7. Invent six practical problems involving subtraction of numbers like those in question six. What is the aim of this exercise? How will you judge the effectiveness of your problems?

8. Secure a dozen specimens of written work of children who are deficient in subtraction. Make a diagnosis of their deficiencies and prescribe corrective practice.

CHAPTER XXII

TEACHING MULTIPLICATION OF WHOLE NUMBERS

ORAL MULTIPLICATION

Introducing the multiplication concept. Teachers find that the idea of multiplication can best be introduced through addition of similar addends. The form 7×3=21 is not as helpful as 7+7+7=three 7's=21. The initial multiplier should not be 1 because 1×5 or 5×1 is very difficult for children. They cannot see the need for multiplying by 1. A common error is 1×5=6. Each number combination should be introduced through practical application. If 4×5 is to be taught, then the problem is to find what four notebooks will cost if one cost 5 cents. Evidently four will cost 5+5+5+5 cents or 20 cents. From this is deduced the expression four 5's are 20 and finally, 4×5=20. If this practice is followed in all the early multiplication combinations, the child will readily grasp the multiplication idea.

Teaching the basic multiplication combinations. *Begin with 2's or 5's.* Some investigators urge that 5 be the first multiplier; others insist on 2 because the child knows 2+2, 3+3, 4+4, 5+5, 6+6, 7+7, 8+8, and 9+9 from his study of the 100 addition combinations. Beginning with 2 as a multiplier does not burden the children with new facts and helps to make the meaning and the language of multiplication clear.

The procedure. The following tabulation may indicate the steps in introducing the initial items of the multiplication tables.

First Step	Second Step	Third Step	Fourth Step	Fifth Step
Initial Concrete Situation	The Additive Form	Multiplication Introduced	Vertical Algorism	Equation or Horizontal Form
One pad costs 2 cents, 2 pads will cost?	2 + 2 = 4 cents	two 2's = 4	$\begin{array}{r}2\\ \times 2\\ \hline 4\end{array}$	2 × 2 = 4
One pad costs 3 cents, 2 pads will cost?	3 + 3 = 6 cents	two 3's = 6	$\begin{array}{r}3\\ \times 2\\ \hline 6\end{array}$	2 × 3 = 6
One pen costs 4 cents, 2 pens will cost?	4 + 4 = 8 cents	two 4's = 8	$\begin{array}{r}4\\ \times 2\\ \hline 8\end{array}$	2 × 4 = 8

[Continue through 2 × 9 or 2 × 12]

Gradation of multiplication combinations. Clapp,[1] who made a very extensive inquiry into the relative difficulty of the various items finds, ×0 and 0× among the most difficult and ×1 and 1× among the most simple. Norem and Knight[2] give the arrangement of the 100 combinations as listed in the table on page 294. The first is the most difficult, the last, the least difficult.

Another gradation, which is simpler but based on teaching experience rather than on scientifically obtained data, is the following:

Teach multiplication first, by 2's, 4's, 5's, 10's through 9 and the reverses.

Teach multiplication, second, by 0's, 1's, 3's, 6's, 7's, 8's, 9's through 9 and the reverses.

And, finally, teach multiplication by 11's and 12's and the reverses.

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[1] F. L. Clapp, *The Number Combinations: Their Relative Difficulty and the Frequency of Their Appearance in Textbooks*, Bulletins No. 1 and 2. Bureau of Educational Research, University of Wisconsin, Madison, Wisconsin, 1924.

[2] Norem and Knight, "The Learning of 100 Multiplication Combinations," *Twenty-ninth Yearbook* of the National Society for the Study of Education, Part II, Ch. vii.

GRADED ARRANGEMENT OF MULTIPLICATION COMBINATIONS

The Combination	The Rank	The Combination	The Rank	The Combination	The Rank	The Combination	The Rank
6 × 9	100	6 × 6	75	7 × 2	50	0 × 5	25
7 × 8	99	9 × 5	74	3 × 5	49	2 × 0	24
7 × 6	98	8 × 9	73	8 × 2	48	1 × 7	23
8 × 6	97	3 × 6	72	2 × 9	47	3 × 1	22
4 × 8	96	8 × 3	71	0 × 4	46	7 × 0	21
4 × 9	95	4 × 4	70	4 × 2	45	1 × 6	20
7 × 9	94	8 × 4	69	0 × 6	44	1 × 5	19
7 × 7	93	9 × 8	68	1 × 1	43	2 × 5	18
6 × 7	92	4 × 3	67	0 × 7	42	7 × 1	17
4 × 7	91	5 × 7	66	2 × 4	41	8 × 1	16
7 × 5	90	9 × 3	65	3 × 3	40	1 × 4	15
5 × 9	89	4 × 5	64	2 × 3	39	5 × 0	14
8 × 8	88	3 × 7	63	9 × 2	38	0 × 8	13
9 × 7	87	6 × 4	62	6 × 2	37	0 × 3	12
8 × 5	86	6 × 3	61	3 × 0	36	4 × 0	11
9 × 6	85	6 × 5	60	2 × 7	35	4 × 1	10
3 × 9	84	3 × 4	59	1 × 0	34	2 × 6	9
8 × 7	83	5 × 5	58	8 × 0	33	0 × 0	8
4 × 6	82	9 × 9	57	3 × 2	32	1 × 9	7
5 × 6	81	5 × 3	56	2 × 8	31	1 × 3	6
9 × 4	80	5 × 4	55	0 × 2	30	5 × 1	5
5 × 8	79	0 × 1	54	0 × 9	29	6 × 1	4
3 × 8	78	7 × 3	53	2 × 1	28	9 × 0	3
6 × 8	77	5 × 2	52	2 × 2	27	9 × 1	2
7 × 4	76	6 × 0	51	1 × 2	26	1 × 8	1

This gradation assumes that the multiplication combinations will be taught in three semesters. In the second semester much is listed, but much of it is mere repetition. Thus, in the multiplication by 7's, the child finds that he learned in the previous grade 7×2, 7×4, 7×5, 7×10 and now must learn only 7×0, 7×1, 7×3, 7×6, 7×7, 7×8, 7×9. In all ×0 and ×1 combinations, only two basic facts must be learned, namely, that in ×1, the multiplicand remains the same and that in ×0 the product is always zero. Hence the work of the second semester is reasonable in quantity.

From analyses of children's results it seems that the size of a product is an index of the difficulty of a combination; that when one combination is wrong, others closely related to it are likely to be wrong; that 91 per cent of errors in multiplication combinations are answers that are correct products for other combinations.[3]

Suggestions for teaching the combinations. 1. Do not teach all the items of a table seriatim. The classroom techniques of an older day sanction the presentation of carefully printed charts that exhibited in vertical arrangement all the "seven times" combination, 7×1, 7×2 . . . 7×12. The number of symbols and figures, if nothing else, make for discouragement and confusion.

2. Make each item the result of an objective process. With the initial combinations the objective procedure is naturally slower and fuller. As the child's stock of known combinations grows, the objective demonstration may be reduced considerably. If the child really knows his addition combinations much time will be saved in teaching the multiplication combinations.

3. Teach the reverse with each combination. When the child learns 4×6 he is ready for its commutation, 6×4. These two forms are learned best when learned together.

4. Introduce the related division form as early as possible. A multiplication combination is really not known until its implications for division are also learned. Suggested forms are given below:

The Known Facts	The Inferred or Derived Facts		
5 × 6 = 30	30 is made up of ..... 6's	30 is made up of ..... 5's	There are ..... 5's in 30
3 × 6 = 18	18 is made up of ..... 6's	18 is made up of ..... 3's	There are ..... 3's in 18
4 × 8 = 32	32 is made up of ..... 8's	32 is made up of ..... 4's	There are ..... 4's in 32
, etc.	, etc.	, etc.	, etc.

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[3] Norem and Knight, *loc. cit.*

The Known Facts	The Inferred Facts				New Forms to be Learned Later	
3×9 = 27	3×? = 27	?×9 = 27	..... 9's = 27	..... 3's in 27	27 ÷ 3 = 9	27 ÷ 9 = 3
5×8 = 40	5×? = 40	?×8 = 40	..... 5's = 40	..... 8's in 40	40 ÷ 5 = 8	40 ÷ 8 = 5
, etc.	, etc.	, etc.	, etc.	, etc.	, etc.	, etc.

5. Mastery of additive counting, that is, learning to count additively by heart, 4, 8, 12, 16, 20, 24, etc., encourages counting when a product of two digits is needed. Hence this form of counting, which serves merely to introduce a new combination, is not repeated after the initial step in a lesson on an item in the multiplication table.

6. Mnemonics are helpful to some children; to others they are retarding factors in learning. They should be taught discriminatingly and only to superior children. The most common mnemonics are the following:

a. When either multiplicand or multiplier is 0 the product is zero.
b. When either multiplicand or multiplier is 1 the product is the other quantity.
c. When either multiplicand or multiplier is 5 the product always ends in 0 or 5.
d. When the multiplier is 11 (through 9×11) the product is the other number repeated, 11, 22, 33, 44, . . . 99.
e. When the multiplier is 10 the product is the multiplicand with a terminal 0.

Steps in teaching a multiplication combination. We must now arrange in sequence the suggestions we have already made for teaching the basic combinations in multiplication.

1. Focalize on one or two combinations and introduce them through practical problems. If 1 pencil costs 4 cents, 6 will cost? One pear cost 4 cents, 6 will cost? One box has 4 crayons, 6 boxes will have?

2. Elicit, six 4's in each case, thus $\begin{array}{r}4\\4\\4\\4\\4\\\underline{4}\end{array}$

3. Elicit the short form of the long addition, six 4's or $\begin{array}{r}4\\\underline{\times 6}\end{array}$

The vertical algorism is introduced. Use perception cards.

4. Teach $\begin{array}{r}4\\\underline{\times 6}\\?\end{array} = 4 \times 6 = ?$ Introduce perception cards.

5. Introduce the commutation $\begin{array}{r}4\\\underline{\times 6}\end{array} = 4 \times 6 = \begin{array}{r}6\\\underline{\times 4}\end{array} = 6 \times 4$

6. Introduce elliptical form of the equation, thus:

$6 \times ? = 24$, $? \times 6 = 24$, $4 \times ? = 24$, $? \times 4 = 24$.

7. Elicit, six 4's in 24; __6's in 24; 24 contains __6's; 24 contains __4's.

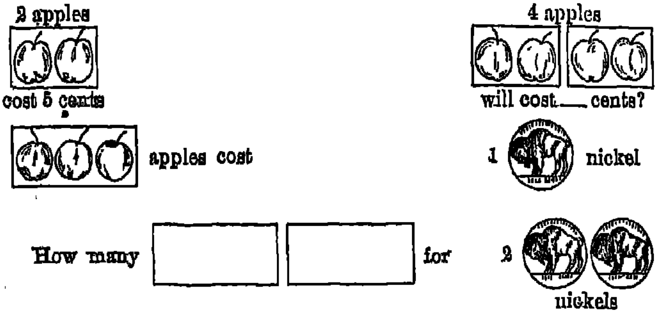
8. As many practical problems as possible, applying 6×4 and 4×6 should accompany each of the above steps.

Each combination is thus taught through a process of measurement and each product becomes replete with ratios.

Ratio analysis as an application of the multiplication facts. With growing mastery of the facts of the multiplication tables children should develop skill in finding values of quantities by ratio and not by unitary analyses. Let us assume that the children know the facts usually referred to as the 2, 4, and 5 times tables. They are then ready to solve the following problems:

Apples cost 2 for 5¢; what will 4, 6, 12 cost?
Apples cost 4 for 10¢; how many can I get for 5 cents? for 20¢?

With the aid of objective material and simple diagrams we can lead children to avoid finding the cost of one. They can easily be lead to understand the following:



As this mode of thought becomes increasingly familiar, objective demonstrations are dropped. Gradually the children acquire helpful control of the ratios in these products of the multiplication tables. Aside from the value of these skills, their application introduces genuine interest in number and provokes intensive thinking.

Increasing the range of the multiplication combinations. As soon as the progress of the child will permit, the combinations in the tables should be carried to higher orders, the products being found by analogy, for example, 3×4=12, 30×4=120, 3×40=120; 5×6=30, 50×6=300, 5×60=300; 7×12=84, 70×12=840; 7×120=840; 8×9=72, 8×90=720, 80×9=720. The child may be told arbitrarily to add the cipher and multiply as if the numbers had no cipher. Such drills, given occasionally, tend to widen the scope of the combinations and thus increase their usefulness.

Miscellaneous drills and reviews in the basic combinations. There is no royal road to mastery of multiplication combinations. Regular and well motivated drills are the price that must be paid. All these suggestions for developing the multiplication facts are designed to reveal the nature of and the constituent factors in the products. These rationalizations must be followed by daily drills that employ the devices enumerated in the addition combinations, namely, the circle, the clock, the ladder, the staircase, the relay race, the railroad ties, etc. The abstract facts like 3×6 and 7×8 must be applied to real problems in the daily practice. An arrangement of numbers like the following may prove helpful:

	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	110	132	144

Here we have regular counting by 2's, 5's, 10's, etc., and the products of all the combinations through 12×12. This arrangement also reveals the peculiar products in the combinations with 5, 10, 9, and 11. Such a table can be used to drill on factors and multiples; it serves to aid children in answering such questions as "What two numbers when multiplied give 36?" Another tabulation can be made by writing the numbers 1 through 12 in the vertical and in the horizontal lines and leaving the boxes blank; the children are then asked to write in the proper space, 24, 48, 36, 18, showing what two numbers will produce them. An interesting series of games may be devised to give zest to these exercises.

WRITTEN MULTIPLICATION

The teaching problems in written multiplication, as in addition and subtraction, arise from inadequate mastery of the facts and techniques suggested for oral work. Responses to the basic number combinations must be prompt and accurate. The written work in multiplication consists essentially of arranging properly the facts learned in the oral exercises.

Multiplying by a single digit. *Gradation.* Ten distinct steps of increasing difficulty must be recognized by the teacher who hopes to lay a sound foundation for more advanced multiplication. Let us analyze these:

The Graded Steps		*The Identification of the New Element*
1. 214 ×2	312 ×3	1. No product over 9; no carrying; no increase in places.
2. 614 ×2	912 ×4	2. No carrying; last multiplication increases the number of places in the product beyond those in the multiplicand.
3. 46 ×2	114 ×4	3. Carrying introduced in one place.
4. 53 ×5	221 ×7	4. Difficulties 2 and 3 combined.
5. 420 ×5	610 ×6	5. Terminal zero difficulty is introduced.
6. 601 ×3	204 ×2	6. Middle zero difficulty shown for the first time, not complicated by carrying.
7. 605 ×3	204 ×6	7. Middle zero difficulty coupled with carrying.
8. 346 ×3		8. The extension to two successive carryings.
9. 300 ×9		9. Two terminal zeros.

10. $\begin{array}{r}406\\ \times 4\\ \hline\end{array}\quad\begin{array}{r}486\\ \times 4\\ \hline\end{array}$ 10. Middle zero with carrying from the first place; carry figure more than one.

This gradation does not imply that ten different types of examples, and therefore ten distinct lessons, must be taught. We have here the difficulties that may beset children. The able pupils can learn more than one type in a lesson or, having learned one, they transfer, in generous measure, both knowledge and skill to the next type in the graded series.

The rationalization. One-digit multiplication may be explained either by an analysis of number or by showing the elements of similarity between multiplication and addition.

1. By analysis To multiply 214 by 2

$$\begin{array}{c}a\\ \begin{array}{r}214\\ \times 2\\ \hline 428\end{array}\end{array}\quad\begin{array}{c}\\ =\\ \\ =\end{array}\quad\begin{array}{c}b\\ \overbrace{\begin{array}{rcrcr}200&+&10&+&4\\ \times 2&&\times 2&&\times 2\\ \hline 400&+&20&+&8=428\end{array}}\end{array}$$

The child is then led to understand that the direct process, under *a* is more economical than the long process under *b*. As a rule, this type of rationalization proves too involved for the young pupil for whom it is intended. For this reason we urge its discontinuance although it is approved by many supervisors.

2. Through addition

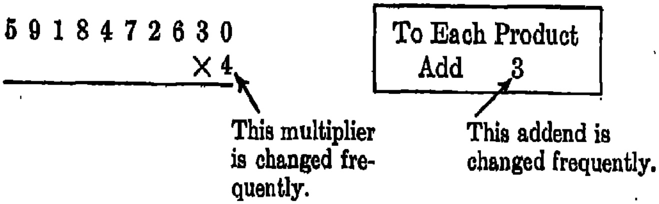
$$\left.\begin{array}{r}214\\ \times 2\end{array}\right\}=\begin{array}{r}214\\ +\,214\\ \hline\end{array}$$

The class is led to add 4 and 4 and conclude two 4's are 8; 1 and 1 are 2, but two 1's are 2; 2 and 2 are 4 but two 2's are four. Both forms are worked out on the board and the children see that the first is shorter and therefore preferable. Similarly,

$$\left.\begin{array}{r}216\\ \times 3\\ \hline 8\end{array}\right|=\begin{array}{r}216\\ 216\\ 216\\ \hline 8\end{array}$$

Children are led to say, "6, 12, 18, but three 6's are 18, put down 8 and carry the 1." (The teacher does so in each case.) "1, 1, 1, are 3 and 1 to carry is 4, but three 1's are 3, and 1 to carry are 4." (Teacher performs the operation in both cases.) "2+2+2 are 6 but three 2's are 6." (Teacher completes process in both cases.) This form of rationalization requires little time, and presents the reason for every step in the process, so that even young children come readily to the conclusion that multiplication is only the short form of addition.

Preparing the pupils for carrying in multiplication. A few days before we expect to teach children multiplication by one digit with carrying, five-minute drills like the following should be given daily:


5 9 1 8 4 7 2 6 3 0
×4
To Each Product
Add 3
This multiplier is changed frequently.
This addend is changed frequently.

The teacher points to *9*, to ×*4*, and to *Add 3*. The response, 36, is rejected because 3 was not added to the product. She then points to *6*, ×*4*, *Add 3* and accepts, not 24, but 27 as the answer. This is repeated four or five times. The multiplier ×*4* and the addend *3* are then changed to other numbers. If this is done regularly, the children acquire the habit of taking two steps—multiplying and adding—in rapid succession. This is precisely the process of multiplying with carrying.

Abstract and concrete numbers. Much has been made of the distinction between abstract and concrete numbers, especially in multiplication. To be sure, the multiplier must always be an abstract number, for how can we multiply by 5 inches, by 12 cents, or by 15 boys? In elementary-school arithmetic this distinction is little more than a bit of pedanticism because the pupils are too young to understand the basic concept. Let us consider these examples:

John has 5 tops, 2 bats, 4 baseballs, and 1 mitt. How many ______ has he? 7 tops + 2 bats + 4 balls + 1 mitt equal 14; but 14 what?

In a class having 15 girls there are twice as many boys as girls. How many boys in the class? How many pupils in the class?

$$\begin{array}{l} 15 \text{ girls} \\ \underline{\times 2} \\ 30 \text{ boys!} \end{array} \qquad 15 \text{ girls} + 30 \text{ boys} = 45 \text{ what!}$$

The fallacies are obvious. In algebra, x represents the *number of*, not the dollars or any concrete things. In each of the two examples, 5, 2, 4, 1, 15 are not the tops nor the balls but the *number of* these concrete things. Hence it is wise to regard numbers in multiplications as abstract and "reason out" the denomination of the product. Thus, "At 8 cents a book, what will 7 cost?" Answer $7\times8=56$, but 56 must have a name; obviously, the 56 must be the number of cents. If there are 15 girls and twice as many boys then $15\times2=30$ but this 30 must refer to boys. 15 girls and 30 boys$=$ $15+30=45$, but this 45 must refer to children, not to boys or to girls. "A floor measuring 30 by 18 feet has what area?" The answer is obtained not by multiplying 30 feet by 18 feet, for that is impossible, but rather by multiplying 30 by 18 and obtaining 540. But this 540 must have a name. The conditions of the example indicate that the number 540 refers to square feet. By treating both multiplicand and multiplier as abstract numbers and relying on common sense for the denomination of the product, the children are freed from a difficult concept relating to abstract multipliers.

Multiplying by two-place numbers. *Multiplying by 10, 20, 40, 60, 70, ... 90.* Before teaching multiplication by two-place multipliers, teach multiplication by 10, 20, 30, ... 90. The process is simple. Children know how to multiply by 7, 5, 8, and 9. Let them treat 10 as if it were a one-digit number. Require the solution of the following:

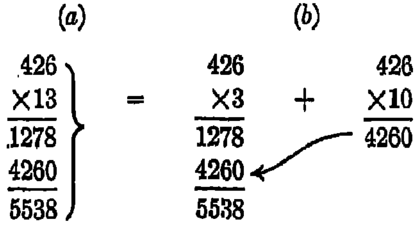
$$\begin{array}{rrrrr} 58 & 172 & 306 & 562 & 91 \\ \times 10 & \times 10 & \times 10 & \times 10 & \times 10 \\ \hline 580 & 1720 & 3060 & 5620 & 910. \end{array}$$

Now ask questions calling for a comparison, in each case, of multiplicand and product. The pupils perceive that in each example the product adds a zero to the multiplicand. Repeat the process, the next day, using 2 and 20, 5 and 50 as multipliers. At the end of this analysis the children will deduce the rule for multiplying by these numbers.

Multiplying by 11 and 12. Many supervisors advise that 11 and 12 should be treated as a two-place multiplier, like 58. In 75×12, the children would multiply by 2, by 1, and then add the two partial products. The advantage, we are told, is that we reduce all two-place multipliers to a uniform process and eliminate ×11 and ×12 from the basic combinations. In the absence of reliable data no final decision is possible. Few intelligent people multiply uniformly with all two-place multipliers. They treat multipliers like 11, 12, 15, 25, and 50 as if they were single numbers. We shall probably do to our pupils as was done to us, because the process we use is the process we like best.

Gradation of two-place multipliers. It is not necessary to postpone two-place multipliers until children have learned all the multiplication steps in one-digit multiplication. To multiply 542 by 13 is simpler than to multiply it by 9 or by 8. The gradation is obvious: (*a*) multiply by 10; (*b*) multiply by 20, 40, 50, . . . 90; (*c*) multiply by 13, 14, 15; (*d*) multiply by 21, 22, 23, 31, 32, 33, 41, 42, 43, multipliers that do not contain numbers above 5; (*e*) multiply by any two-place number.

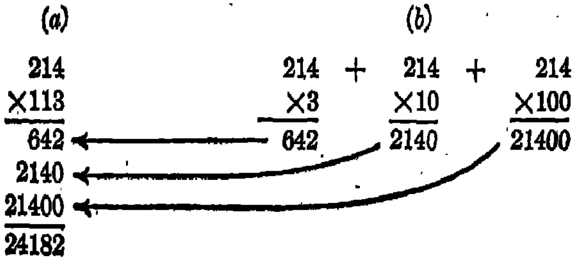
The rationalization. It is not difficult to demonstrate to children that multiplying by 13 is the same as multiplying by 3 and then by 10. It is important to note the order, 3+10, not 10+3. The blackboard arrangement of the work is given below:



The children see that (426×13) = (426×3) + (426×10); that the arrangement under (*a*) contains all that is essential under (*b*), but that it saves much useless writing; that (*a*) is decidedly preferable to (*b*).

The terminal zero in the partial product should be used until the process is thoroughly habituated. Then we can deduce the rule that the extreme right-hand figure of each partial product must be directly under the digit used in the multiplier. We must avoid the direction "skip a place" for it is neither mathematical nor helpful.

Multipliers of three or more places. *The rationalization.* The process with two-place multipliers may be adapted to meet the needs of three-place multipliers. The explanation is easily seen in the following arrangement.



This compact process is merely a repetition of the long process under (*b*).

The long process which is worked out first to prepare for the short arrangement under (*a*).

In the initial stages the ciphers may be introduced, but when the sequence of steps has been grasped by the child the multiplication should proceed by using units, 3—then the tens, a 1—then the hundreds, a 1—as the multipliers, placing the second and the third partial products in the correct relative positions.

Further gradation. The continued gradation is quite obvious. Unusual care must be taken with multiplication in which zeros enter in various places in multiplicand and multiplier. It is well to give practice in the following cases:

204	270	386	380	3800	234	2004	2356	207,	
236	542	540	540	540	206	1236	2004	604,	etc.

Stone[4] reports that 50 per cent of high-school graduates failed in multiplying 53 241 by 20 304. The reason is not difficult to find: inadequate practice in all important types of multiplication.

As in the other fundamental operations, practice must be given in application of multiplication to real problems in making out bills, in computing areas, in ascertaining costs—in any genuine business situation.

Inventory of errors in multiplication and remedial suggestions.

The Error	*The Remedial Measure*
1. Errors due to imperfect mastery of basic combinations, especially with zero and one	1. The corrective measures outlined for similar errors in addition apply here with obvious adaptations. In general the procedure is (*a*) teach; (*b*) diagnostic testing to ascertain individual and group weaknesses; (*c*) differentiated practice; (*d*) retest.
2. Errors due to carrying *a*. Carrying wrong number	2. Adequate practice in carrying must be provided in examples especially devised to introduce the conditions

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[4] John C. Stone, "The Stone Arithmetics," *Teachers' Manual*, Grades 1 to 6 (Benjamin H. Sanborn & Co., 1928), p. 27.

The Error	*The Remedial Measure*
b. Forgetting to carry *c*. Unnecessary carrying	referred to under 2 *a*, *b*, and *c* of the opposite column. Examples like 2132 × 13 and 1324 × 2 should alternate with 6974 × 85 to make sure that carrying does not become an unfailing process in all multiplication.
3. Errors due to zero *a*. in the basic combinations *b*. in terminal positions *c*. in internal positions	3. Children who experience the zero difficulties should be given simple explanations of the effect of a zero in multiplication in one of the three relations referred to in the opposite column. After that individual practice with individual records of progress will bring improvement.
4. Errors due to misplacing of partial products, which may be due to *a*. ignorance, especially with zeros, or *b*. carelessness	4. The cause must be ascertained. If it is ignorance, then the procedure is, of course, reteaching, practice, and retesting. If the cause be carelessness then motivation is essential. The child should be led to realize that he has the necessary knowledge and skills, but that his grades are below his ability because of failure to attend carefully to his work. Genuine motivation, regular inspection of written work, and judicious praise are usually efficacious corrective measures.
5. Errors due to misinterpretation of signs and to lack of knowledge of the technical terms of multiplication	5. In planning the day-by-day work, introduce very gradually but incidentally the signs × and =; distinguish × from +. In wording exercises the term product is taught by asking children to find the products of the following: 508 × 72, etc. In the same way multiplier and multiplicand are presented. No formal definition of these terms should be given or required; accurate identification is quite sufficient.

PROBLEMS FOR STUDY AND DISCUSSION

1. Give your judgment, with adequate support, of the validity of the following statements:
   - (*a*) It is desirable to begin multiplication with 2 as the multiplier.
   - (*b*) The multiplication combinations should not be taught seriatim.
   - (*c*) Mnemonics are of extremely doubtful value to most children.

2. Give a teaching application of each of the following statements:
   - (*a*) The measurement idea should prevail in teaching multiplication combinations.
   - (*b*) Most products of the multiplication combinations can be found by the children themselves.
   - (*c*) The child who knows 6 × 4 does not necessarily know 4 × 6.
   - (*d*) The range of the multiplication combinations can readily be increased.

3. Evolve a suitable gradation for the zero difficulty in multiplication. State the steps and identify the new element in each.

4. Examine a set of children's arithmetic papers in which multiplication examples are stressed. What types of errors do you find? What is the relative frequency of each? How does your list compare with the list of common errors found in the closing part of this chapter?

5. For each of the five most common errors, indicate suitable corrective measures.

6. Examine the sections on multiplication in a textbook in arithmetic and in a practice or workbook. Note among other things, (*a*) the gradation; (*b*) the illustrative aids; (*c*) the type of drill; (*d*) the application of number combinations to real situations. To what extent are the basic principles elucidated in this chapter carried out? What suggestions for procedure did you find in this examination that are not included in this chapter?

CHAPTER XXIII

TEACHING DIVISION OF WHOLE NUMBERS

Oral Division

Introducing the concept of division. The limited vocabulary of young children is accountable, in great measure, for the difficulty they have in recognizing the function of division. They are baffled by the expressions "7 goes into 42 how many times?" "42 contains 7 how many times?" "42÷7"; "7 into 42," and the like. The idea of division must be introduced when the multiplication facts are being learned. Thus the child sees : : : and concludes three 2's make 6, 3×2=6. From this he can be helped to discover how many 2's in 6, or in 4. The following sequence is helpful in teaching combinations like, 21÷7 and 21÷3:

a. 1 book costs 3¢; 7 books cost 21 cents: there are ____7's in 21; there are ____3's in 21.

b. 1 book costs 3¢; how many books can I buy for 21 cents: there are ____3's in 21; 21 ÷ 3 = 7.

1 pencil costs 7¢; how many pencils can I buy for 21 cents: there are ____7's in 21; 21 ÷ 7 = ?

After much patient repetition of the patterns under *a* and *b*, the concept *divide* takes on real meaning. The conclusion is quite obvious: all division combinations must be taught through concrete situations.

Division a test of mastery of fundamentals. In division the child must fall back on the combinations in the other fundamental operations. Osburn[1] lists 1237 items that must be known before division can be mastered. This list presents a more discouraging picture than actual facts warrant. It assumes that every one of these facts constitutes a separate bond calling for independent learning. While we do not know the extent of transfer of one number fact to related number facts, we have abundant evidence that such transfer does take place. In division the worker's proficiency is determined by his ability to multiply and subtract. Thirty per cent of the errors in division have been traced to lack of knowledge of multiplication combinations. In division we assume that the child can multiply, but the arrangement is now wholly changed. Let us examine two arrangements in which 37 is multiplied by 7. In *a* multiplicand, multiplier, and product are in their accustomed places; in *b* the arrangement is quite different and certainly less helpful. Here is a cause of children's difficulty that calls for patience in the slow learning of division.

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[1] W. J. Osburn, *Corrective Arithmetic* (Houghton Mifflin Co., 1924), p. 21.

a

$$\begin{array}{r} 37 \\ \times 7 \\ \hline 259 \end{array}$$

b

$$\begin{array}{r} 7 \\ 37\overline{)259} \\ 259 \\ \hline \end{array}$$

The number combinations in division. The basic combinations of division, 36÷4, 36÷9, 48÷6, 48÷8, etc., are taught with the multiplication combinations. We saw that the child learns 5×9=45, then 5×?=45, and finally ___ 9's in 45. One more step brings the child to the inescapable form, 45÷9 and 45÷5. The difficulty, it was shown, is one of language, not arithmetic. Our chief concern in teaching the combinations of division is to make the division idea clear to the child. As in multiplication, the 0 and 1 combinations are very troublesome; the quotients of 1÷1 and 9÷9 are frequently given as 0; 0÷5, as 5; 5÷1, as 1. Patient drill and repeated demonstrations ultimately bring correct responses.

Written Division

Dividing by a digit. *Gradation.* Fine gradation is more important in division than in many other operations. Let us analyze a system of gradation designed to quicken the learning process.

The Step	*The Progression Identified*
1. $2\overline{)68}$, $3\overline{)69}$, $4\overline{)84}$, $3\overline{)696}$, $2\overline{)468}$	1. Two- and three-place quotients; no carrying; no remainders.
2. $2\overline{)69}$, $3\overline{)67}$, $4\overline{)86}$, $3\overline{)695}$, $2\overline{)467}$	2. Same as 1 but with remainder.
3. $2\overline{)56}$, $3\overline{)75}$, $4\overline{)52}$, $3\overline{)675}$, $2\overline{)456}$	3. Carrying of 1 from tens' to units' place but without remainder.
4. $3\overline{)84}$, $4\overline{)68}$, $3\overline{)652}$	4. Carrying increased above 1.
5. $3\overline{)85}$, $4\overline{)69}$, $3\overline{)651}$	5. Steps 2, 3, and 4 combined.
6. $3\overline{)435}$, $4\overline{)652}$	6. Two successive carryings with or without remainder.
7. $2\overline{)126}$, $3\overline{)246}$, $4\overline{)368}$	7. The difficulty arising when the initial number of the dividend is smaller than the divisor.
8. $2\overline{)570}$, $3\overline{)960}$, $3\overline{)970}$	8. The terminal zero in the dividend.
9. $2\overline{)408}$, $2\overline{)508}$, $3\overline{)609}$, $3\overline{)708}$	9. The internal zero in the dividend.
10. Divisors increase to 7, 8, 9.	10. Difficulties listed above introduced with increasing divisors.

A careful adherence to a gradation as fine as this will do much to simplify the process for the learner and help him materially to acquire the skill necessary for success with divisors of two or more places.

Rationalization. The mode of explanation of the process is shown in outline form.

1. 64 ÷ 2 (64 = 6 dimes and 4 cents)

$$a.\ \begin{array}{r@{}l} & \begin{array}{cc} 3 & 2 \end{array} \\ \cline{2-2} 2\Big) & \begin{array}{cc}\text{dimes} & \text{cents}\\ 6 & 4\end{array}\end{array} = b.\ \begin{array}{r@{}l} & 32 \\ \cline{2-2} 2) & 64\end{array}$$

It should be made clear that the short form repeats the long process.

2. 65 ÷ 2 (65 = 6 dimes and 5 cents)

$$\begin{array}{r|ll} & 3 & 2 \text{ and 1 cent left over.}\\ \hline a.\ 2\,) & \text{dimes} & \text{cents}\\ & 6 & 5 \end{array} \qquad = b.\ \begin{array}{l} \quad 32 \text{ and 1 cent left over.}\\ 2\overline{)65}\end{array}$$

The children are told to express the remainder in the form used above or in the form, "remainder over divisor." No reason need be given. The two forms of writing the remainder should be taught in the primary grades, but not together.

3. 74 ÷ 2 (74 = 7 dimes and 4 cents)

$$\begin{array}{r|ll} & 3 & 7\\ \hline a.\ 2\,) & \text{dimes} & \text{cents}\\ & 7 & 4 \end{array} \qquad = b.\ \begin{array}{r} 37\\ 2\overline{)74}\end{array}$$

2 into 7 dimes = 3 and 1 dime over. Change the 1 dime into cents. 10 cents and 4 cents = 14 cents. 2 into 14 cents = 7 cents.

Again, it must be made clear that the short form repeats all the steps of the long form.

No rationalization should be given beyond that offered for the division by two. The process should be reduced to habit as quickly as possible through repeated solutions of a large number of graded examples. In the example $3\overline{)1755}$ the children learn to think, "3 into 17" and to say "5"; to think, "3 into 25" and to say "8"; to think, "3 into 15" and to say "5." As they say a number they write it in its correct place.

No auxiliary figures should be permitted with divisors that are small, say, 2, 3, 4, 5, and 10. With larger numbers, 6, 7, 8, 9, 11, and 12, children often do more rapid and more accurate work when they follow the algorism for long division, writing down all the multiplications and subtractions. Teachers should encourage mechanical operations *without* auxiliary figures shown in the accompanying solution, But, in the case of nervous children, rate and accuracy are both reduced by

$$\begin{array}{r} 0652\\ 9\overline{)5\overset{4}{8}\overset{1}{6}8}\end{array}$$

denying them the use of the crutches 0 in the quotient and the small 4 and 1 in the dividend. Such pupils should not be made victims of fixed pedagogical procedures. Auxiliary figures in multiplication are not of any measurable worth; in division and in subtraction they contribute materially to the confidence of some types of workers.

Division by a number of two orders. *Gradation.* Instead of using the sequence 11, 12, 13, 14, 15 . . . 19, 21, 22, 23, 24 . . . 28, etc., as the numerical order of divisors, it has been found more advantageous to use **11, 12, 19, 21, 29, 31, 39, 41, 49** first and then, any number like, 24, 36, 47. This sequence is justified by the fact that these divisors in heavy type are so close to 10, 20, 30, 40, and 50, respectively, that they present less difficulty to the children when they seek an approximate or trial quotient figure. This technique will be explained in greater detail later in the discussion.

A second factor in gradation at this point is the selection of the dividend. The sequence suggested, therefore, is:

$$21\overline{)231} \qquad 21\overline{)294} \qquad 21\overline{)105} \qquad 21\overline{)1071}$$

instead of using $21\overline{)1846}$ as the first type of example. The reason for this general gradation will become apparent to the reader after he solves each example.

Rationalization. No attempt should be made to give explanations or justifications for the procedure in this more advanced division. The children should learn it by continuing the process followed in the simpler division. The old distinction between short and long division, non-existent in mathematics, with changes in algorism, should not be perpetuated. To teach children to divide by 19, 21, 29, 31, and 39, we continue the process begun earlier in dividing by a single digit divisor. To illustrate: the child divides 278 by 4, thus: $\begin{array}{r} 69\frac{3}{4} \\ 4\overline{)278} \end{array}$; 4698 by 5, thus: $\begin{array}{r} 939\frac{3}{5} \\ 5\overline{)4698} \end{array}$. But when the divisors are raised to 8, 9, or 12 it is necessary, in the interests of accuracy and speed, to permit the following form instead of the shorter one.

Long Form

```
    733 4/8
 8)5868
   56
   --
    26
    24
    --
     28
     24
     --
      4
```

Short Form

```
    733 4/8
 8)5868
```

Long Form

```
     489
12)5868
   48
   --
   106
    96
   ---
    108
    108
    ---
```

Short Form

```
     489
12)5868
```

The children are shown that in the long form the carrying, multiplying, and subtracting are performed on paper rather than "all kept in mind." The divisor may now be increased to 21 and the child is shown that the process does not vary, for example,

```
     11                         279 9/21
 21)231          or         21)5868
    21                          42
    --                          --
     21                         166
                                147
                                ---
                                 198
                                 189
                                 ---
                                   9
```

The algorism in all division should remain uniform. No adequate reason has, as yet, been adduced for retaining the distinction between short and long division. To divide by 9 may be more difficult than to divide by 20. After months of work with large divisors children become proficient in division and will, of their own accord, stop writing the multiplications and subtractions when dividing by 8 or 9.

Those who favor the retention of the two algorisms[15] argue that business people prefer the short division arranged thus, $6\underline{)432}$; that the algorism $6\overline{)432}$ is inconvenient in finding averages or in successive division as is shown in the accompanying column; that no confusion has ever arisen in the minds of children between long and short divisions. In answer, we may submit that business people do not concern themselves with such school matters; they ask for correct answers in reasonable time and leave all questions of methodology to the school. The long division arrangement in the two instances cited is not convenient in finding averages or in successive divisions; but it is better to rewrite the dividends, 185 of *a* and 978 of *b* than to continue the useless distinction in all division. The fact that the two forms, long and short division, were learned in former generations is no justication for continuing them.

a

$$\begin{array}{r} 36 \\ 46 \\ 64 \\ 39 \\ 4\overline{)185} \\ \hline 46.25 \end{array}$$

b

$$\begin{array}{r} 6\underline{)5868} \\ 2\underline{)978} \\ 489 \end{array}$$

Trial quotients. A real difficulty is experienced when children try to find a quotient digit. They proceed by trial and error. If 4 is too small, then they select 5, 6, and 7 in succession and finally find that 8 is the correct quotient digit. By this time their continuous erasures have gone through the paper. The waste in such unguided trials must be checked.

Graded practice in finding trial quotients must be provided. An arrangement like that found on page 316 is a good introductory device.

The chart suggested on page 299 for teaching multiplication combinations can be used with good effect. "How many

---

[15] David Eugene Smith and John C. Stone are among them.

Divisor 4

4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31
32	33	34	35
36	37	38	39

Divisor 7

7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69

Teacher points to 33, child sees 32, says 8 and 1 over.

Teacher points to 27, child sees 24, says 6 and 3 over.

Teacher points to 34. Child sees 28, says 4 and 6 over.

Teacher points to 60. Child sees 56, says 8 and 4 over.

8's in 65," is answered by finding the nearest multiple of 8, that is, 64. This is in column 8, hence, the answer, "8 and 1 over," is readily suggested.

A preparation for finding the quotient digit when dividing by a two-place divisor is to use such numbers as 19, 21, 29, 31, 39, 41, 49 in the early lessons. The children are then told to regard 19 and 21 as 20 for trial quotient purposes; 29 and 31 as 30; 39 and 41 as 40; 49 and 51 as 50. This reduces the random trials.

$$\begin{array}{r} 2 \\ 19\overline{)5476} \end{array}$$

Think of 19 as 20 and 54 as 50. Then say, "20 into 50 = 2."

$$31\overline{)9462}$$

Think of 31 as 30 and 94 as 90. Then say, "30 into 90 = 3."

$$39\overline{)9462}$$

Think of 39 as 40 and 94 as 90. Then say, "40 into 90 = 2."

There is no assurance that the first trial quotient number will always be correct, but it will generally be not more than one from the true quotient digit. The whole example need not be done to give practice in finding the trial quotient number. Charts with suitable practice material should be prepared for daily five-minute oral drills until the children give evidence of ability to make a sensible selection.

$21\overline{)130}$ $21\overline{)160}$ $21\overline{)180}$ $21\overline{)190}$ $21\overline{)210}$

$19\overline{)130}$ $19\overline{)160}$ $19\overline{)180}$ $19\overline{)190}$ $19\overline{)210}$

$49\overline{)150}$ $49\overline{)250}$ $49\overline{)200}$ $49\overline{)350}$ $49\overline{)458}$

$49\overline{)140}$ $49\overline{)259}$ $49\overline{)143}$ $49\overline{)160}$ $49\overline{)361}$

Simple rules for finding the trial quotient digit may prove helpful and interesting. Let us inspect the following examples:

$$\begin{array}{r} 3 \\ 54\overline{)17896} \\ \underline{162} \\ 16 \end{array}$$

Try 5 into 17, not 54 into 178; answer, 3.

$$\begin{array}{r} 4 \\ 43\overline{)17896} \\ \underline{172} \\ 6 \end{array}$$

Try 4 into 17, not 43 into 178; answer, 4.

This practice is especially helpful with two-place divisors when the number in the units' place is less than 6.

Now let us analyze these examples:

$$\begin{array}{r} 4 \\ 58\overline{)25986} \\ \underline{232} \\ 27 \end{array}$$

Try, not 5 into 25, but *6* into 25; answer, 4.

$$\begin{array}{r} 7 \\ 58\overline{)45872} \\ \underline{406} \\ 52 \end{array}$$

Try, not 5 into 45, but *6* into 45; answer, 7.

$$\begin{array}{r} 5 \\ 67\overline{)35167} \\ \underline{335} \\ 16 \end{array}$$

Try, not 6 into 35, but *7* into 35; answer, 5.

When the unit digit is 6 or higher, then increase the tens' number by 1. This we did in the three examples above. While these two simple devices may be helpful, trial quotients require, in the last analysis, an approximation based on reasoning.

The Austrian method of division. Austrian subtraction has a parallel in division. In Austrian division additive and multiplicative processes are emphasized.

$$\begin{array}{r} 421 \\ 326\overline{)137246} \\ \underline{1304} \\ 684 \\ \underline{652} \\ 326 \\ \underline{326} \end{array}$$

Old or usual form of division with remainder emphasized throughout the process.

$$\begin{array}{r} 421 \\ 326\overline{)137246} \\ 684 \\ 326 \end{array}$$

Austrian method of division. Process: 4 × 6 = 24; what added to 4 gives a 2? An 8; put down 8 and carry the 3, 24 + 8 = 32.

4 × 2 = 8; 8 + 3 = 11; what added to 1 gives a 7? A 6; put down 6 and carry the 1, 11 + 6 = 17.

4 × 3 = 12; 12 + 1 = 13; what added to 13 gives 13? Answer, 0, hence no other number need be put down.

Now bring down the 4 of the dividend; 326 into 684, 2.

2 × 6 = 12; what number added to 2 gives a 4? Answer, 2; put down the 2; carry 1, 12 + 2 = 14.

2 × 2 = 4; 4 + 1 = 5; what added to 5 gives an 8? Answer, 3. Put down 3; nothing to carry 2 × 2 = 4, 4 + 1 = 5.

2 × 3 = 6; what added to 6 gives a 6? Answer, 0.

Now bring down 6; 326 into 326, 1 and process is completed.

Division by the Austrian system requires less writing and combines both multiplication and subtraction. Whether this leads to economy of time and an increase in accuracy remains to be demonstrated by actual achievement. Writing fewer figures and combining two processes may not only require more mental effort and more time on the part of the child, but may also cause a greater number of errors in computation.

Quotition and partition. Division may take one of two forms. Let us assume that a given whole must be divided into a number of groups. It may become necessary to find the number of units in each group. If 16 cents is to be divided among a number of children, each receiving 2 cents, then it is necessary to find the number of children who will be given 2 cents each; but if 16 cents is to be distributed among 8 children, then it is necessary to find the number of cents that will be given to each child. These two forms of division are known as *quotition* and *partition,* respectively. In *quotition* the number of equal groups is sought; the numbers in the dividend and in the divisor refer to the same commodity; the numbers in the dividend and in the quotient refer to different commodities. In *partition* the number of units in each equal group is sought; the number in the dividend and in the quotient refer to the same commodity; the number in the divisor is abstract. Partition is a form of division closely related to finding fractional parts of a number, for example, "1/5 of $15"; quotition is a form of division closely related to problems which find what fractional part one quantity is of another, for example, "$3 is what part of $15?" While these two forms of division may interest the teacher, the classification should never be taught to the children.

The more complicated forms of division. The very difficult divisors have not been considered here. Careful gradation will tend to make them less troublesome and constant practice will help to insure mastery. Both the gradation and the drills for the advanced forms of division can be found in any good textbook in elementary arithmetic. The introduction of zeros in the divisor or the dividend, for example, $204\overline{)76\,058}$, does not complicate the process as it does in multiplication. If the type of divisor that ends in one or more zeros is postponed until decimal processes are understood, the very zeros will simplify the division. Instead of dividing $2478.54 by 800, the decimal operation may be introduced, and the divisor may be reduced to 8, for example, $8.00\overline{)\$24.7854}$, thus saving time and effort.

Signs and terms in fundamental operations. There are important signs and terms, the significance of which children must learn. Chief among the former are +, −, ×, ÷, =; among the latter, *sum, minuend, subtrahend, difference, multiplicand, multiplier, product, dividend, divisor, quotient, unit,* and *remainder.* The meaning and the function of each should be learned incidentally in the course of solving examples. Children should not be required to memorize set definitions of any of these. Ability to illustrate and recognize each of these terms and to perform the process indicated by each of these signs should be regarded as sufficient mastery.

Inventory of errors in division and remedial corrections. We must now list the common errors that children make in division and indicate corrective measures for each. We are not seeking a complete list of errors—merely those made by considerable groups of children in any school.

The Errors	*The Remedial Measures*
1. Errors due to inadequate mastery of basic combinations *a.* $\begin{array}{r} 8 \\ 4\overline{)36} \end{array}$ *b.* $\begin{array}{r} 1 \\ 5\overline{)1} \end{array}$ *c.* $\begin{array}{r} 0 \\ 6\overline{)6} \end{array}$ *d.* $\begin{array}{r} 6 \\ 6\overline{)0} \end{array}$ *e.* $\begin{array}{r} 1 \\ 1\overline{)6} \end{array}$	1. Division combinations are usually not stressed as much as multiplication combinations. We must compensate for this oversight by giving diagnostic tests and prescribing individual and differentiated drill material according to the needs revealed. Special attention must be given to the 0 and 1 combinations. With bright children a careful explanation and concrete demonstration of $6\overline{)6}$, $6\overline{)1}$, $6\overline{)0}$ will go far to eliminate the errors listed in the opposite column. Combinations in which a number is divided by itself also need special attention.

The Errors

The Remedial Measures

The process, therefore, is, (*a*) test; (*b*) reteach and provide differentiated drill; (*c*) retest.

2. Errors due to carrying

3. Failure to use the first figure or figures when it or they are less than the divisor

2, 3. *a.* Careful gradation of the various types in division. Provide ample practice for each type.

b. Simple rationalization for all who can understand it.

c. Use mechanical devices, thus—"There must be a quotient digit for every digit in the dividend."

To illustrate:

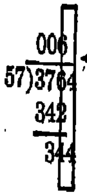
$$\begin{array}{r} 0132 \\ 9\overline{)1188} \\ \underline{9} \\ 28 \\ \underline{27} \\ 18 \\ \underline{18} \end{array} \quad \begin{array}{r} 0074\frac{4}{16} \\ 16\overline{)1188} \\ \underline{112} \\ 68 \\ \underline{64} \\ 4 \end{array} \quad \begin{array}{r} 0358\frac{9}{24} \\ 24\overline{)8601} \\ \underline{72} \\ 140 \\ \underline{120} \\ 201 \\ \underline{192} \\ 9 \end{array} \quad \begin{array}{r} 0303\frac{12}{31} \\ 31\overline{)9405} \\ \underline{93} \\ 105 \\ \underline{93} \\ 12 \end{array}$$

It is better to permit the crutch of zeroes to the left of the quotient than have initial dividend figures neglected.

4. Errors due to misplacement of quotient digits and dropped zeros

$$\begin{array}{r} 3\ \ 49 \\ 7\overline{)21343} \end{array}$$

$$\begin{array}{r} 304\frac{6}{7} \\ 7\overline{)21343} \end{array}$$



← Child sees that next quotient digit must be placed directly over the units' place of the dividend.

4. *a.* Explain importance of careful placing of quotient digits. Show how arduous work is lost by this form of carelessness. Motivation is a great aid.

b. Give practice in copying completed examples in which quotient digits are very accurately placed.

c. Use mechanical device. Before placing a quotient digit, each child should find the alignment with a narrow strip of cardboard.

The Errors	*The Remedial Measures*
5. Errors due to interchanging dividend and divisor, especially when the dividend is smaller than the divisor. Thus $3 \div 6$ is erroneously turned into $3\overline{)6}$	5. *a.* Carry out suggestions for giving children an understanding of the division idea. *b.* Vary relative size of dividend and divisor: Had 8 cents to give to 2 boys. What does each get? Had 4 dollars to give to 8 boys. What does each get?
6. Errors due to carelessness: omitting the remainder; mis-reading signs, ÷ for — and vice versa; bringing down wrong digit from the dividend to continue the division	6. *a.* For a time, give no credit for work that is correctly conceived but is terminated by wrong answers. Do not give partial credit. *b.* Strong motivation. Let the children see what is at stake. Make clear to them that they know the process but do not apply what they have mastered. *c.* Judicious praise at signs of improvement.

Problems for Study and Discussion

1. Give two moot questions that arise in the teaching of division. What are the opposing arguments? Which are based on opinion and which on fact?

2. Analyze the processes in division of whole numbers and indicate what can be and what cannot be rationalized in teaching fourth-grade children?

3. Set forth a series of four graded steps designed to help pupils find trial quotient digits.

4. Administer a diagnostic test in division to children in a suitable grade. What are the most common errors? What is the relative frequency of these errors? Compare the list of errors you derive with the list of errors in the last section of this chapter. To what extent do the two lists agree?

5. List all the difficulties that zero may cause children in division. What teaching procedures will minimize each of these difficulties?

6. Devise twelve problems that require division. In terms of what standards do you accept or reject a problem suggested by a classmate or colleague?

CHAPTER XXIV

TEACHING CHECKS, PROOFS, AND SHORT CUTS

CHECKS AND PROOFS

Importance of checking results. Children must be impressed with the absolute standard of accuracy demanded by the business world. To encourage the child the school often gives credit for correct planning of a solution even though the correct answer is not obtained. Children may therefore underestimate the importance of the answer in arithmetic. Appropriate forms of checking results must be embodied in the fundamental processes. The consistent use of checks and proofs inculcates a critical attitude toward one's own work. They teach children the art of self-correction; they diminish the number of absurd answers and stress accuracy in arithmetical computations.

Methods of checking and proving. We may check results either by obtaining an approximate answer or by working out a full proof.

By approximations. When, for some reason, it is impossible or impractical to prove the answer absolutely, we may merely test its reasonableness. Thus in the example, "Find the interest on $3504.27 for 3 yr. 5 mo. at 3%," endless and difficult computations are required to prove the answer by "working it backwards." In such situations we assume that the time is 3 yr. and 4 mo. or 3⅓ yr. and that the principal is $3500, not $3504.27. The approximate interest is therefore 10 per cent of $3500, or $350. It is obvious that $350 or any sum less than $350 must be incorrect. The answer that is a little above $350 is reasonable and its correctness is assumed. This method of approximation demands sustained thought and tends to eliminate ludicrous answers.

By exact proofs. There is a variety of methods of testing for the absolute accuracy of the result. These are listed below with appropriate illustrations.

1. Reword the problem, making the element to be found a given or a known element. Then, solve the new example to ascertain whether the figure found is identical with the figure sought.

Illustration: A plot of land mortgaged for 60% of its value is mortgaged for $3600. What is the land worth? *Ans.* $6000.

Check: Land worth $6000 is mortgaged for $3600. What per cent of the value of the land is covered by the mortgage?

$$\frac{3600}{6000} = \frac{3}{5} = 60\%$$

2. Ascertain whether the answer meets the conditions of the problem.

Illustration: A retail dealer who usually receives a discount of 15% finds that goods listed for $350 were charged at $280. Is the retail merchant being allowed a better discount? If so, how much does he gain? Usual discount = 15% of $350 = $52.50. New discount, $70, minus usual discount, $52.50 = $17.50, increase discount.

Test. Usual discount $52.50 + new increase $17.50 = $70 new discount. $\frac{70}{350} = \frac{1}{5} = 20\%$ = new rate, hence 5% increase. 5% of $350 = $17.50 = increase discount.

3. Solve by another method.

Illustration: Goods marked $36.60 per gross are sold to Mr. A at a discount of 25% and 10%. What is the final cost per gross to Mr. A?

First Method: 100% − 25% = 75%; 10% of 75% = 7.5%, hence final cost is 67½% of $36.60 or $24.71.

Second Method: $\frac{3}{4}$ of $36.60 = $27.45; $\frac{9}{10}$ of $27.45 = $24.705.

4. Rework the example with great care. Very often another form of solution involves an inordinate amount of computation. Working the example or problem backward is even less

practical. In such cases a careful examination of every step in the solution is the only reasonable check that may be made.

5. Check the mechanical processes involved.

Checks for addition:

a. Add the columns in reverse order. If the columns were added up, add them down, or vice versa. If the answers agree, the assumption is that the final result is correct.

b. Successive subtraction. From the sum take, in successive subtractions, each of the addends. This method is too long to be considered seriously.

c. Casting out nines.

542 add 5 and 4 and 2; 11;	cast out 9 of 11, result 2	Add these; 13; cast out 9 of 13; result is 4.
376 add 3 and 7 and 6; 16;	cast out 9 of 16, result 7	
489 add 4 and 8 and 9; 21;	cast out 9's of 21, result 3	
757 add 7 and 5 and 7; 19;	cast out 9's of 19, result 1	
2164 add 2 and 1 and 6 and 4; 13;	cast out 9 of 13, result 4	

When the final numbers agree, as 4 and 4, the result is assumed to be correct. While this is interesting to number minded children, the process should not be taught for reasons given under *d*.

d. Reducing to single digit.

542 add 5, 4, 2; result 11; add 1 and 1;	result 2	Add 2, 7, 3, 1; result 13; add 1 and 3, result 4.
376 add 3, 7, 6; result 16; add 1 and 6;	result 7	
489 add 4, 8, 9; result 21; add 2 and 1;	result 3	
757 add 7, 5, 7; result 19; add 1 and 9; 10; Now add 1 and 0;	result 1	
2164 add digits of the answer; $2+1+6+4=13$; $1+3=4$		

The last two methods (*c* and *d*) are too long to merit practical application. They are mere reciprocals of each other. Assume that the answer was 2146, not 2164, and apply either mode of test; 2146 would give $2+1+4+6=13$; cast out 9 or reduce to a digit and 4 is obtained. 2146 answers the test but is incorrect. These two modes of checking are, therefore, interesting historically but are not reliable.

e. Add groups of addends and then find the total result of these partial sums.

354 }
273 } A
856 }
977 }
522 } B
389 }
33
3371

Add all of A
354
273
856
1483 → Add 1483
and 1888
3371 Check

Add all of B
977
522
389
1888 ↗

This is obviously a check, but the time involved makes its use unwise.

f. Re-add but do not carry figures to next order.

354
273
856
11
1483 check this thus,

354
273
856
13 total of units' column
17 total of tens' column
13 total of hundreds' column
1483 check

This is like the method suggested under *a*.

g. Add all but one addend. To the sum add the addend not included. The total sum is the check number.

354 }
273 }
856 } A
977 }
522 }
389 B
3371

A { 354
273
856
977
522
2982
B = 389
3371 check

In the final analysis, the first and the last two suggestions are the only useful ones; few practical workers have recourse to plans *b*, *c*, *d*, and *e*.

Checks for subtraction:

a. Add the answer to the subtrahend to obtain the minuend.

b. Subtract the answer from the minuend to obtain the subtrahend.

Checks for multiplication:

a. Interchange multiplier and multiplicand to obtain the same product. Similar to *a* under addition.

b. Divide the product by either multiplicand or multiplier to obtain the other factor. This method requires too great an expenditure of time.

c. Casting out nines or reducing to a digit.

$$\begin{array}{l}
\left.\begin{array}{rl} 254 & \text{add } 2, 5, 4;\ 11;\ \text{cast out } 9;\ 2 \\ \times\ \underline{58} & \text{add } 5, 8 \quad\ 13;\ \text{cast out } 9;\ 4 \end{array}\right\} 4 \times 2 = 8,\ \text{cast out } 9;\ 8 \\
2032 \\
\underline{1270} \\
14\,732\ \text{add } 1, 4, 7, 3, 2, 17;\ \text{cast out } 9;\ 8
\end{array}$$

$$\begin{array}{l}
\left.\begin{array}{rl} 254 & \text{add } 2, 5, 4;\ 11;\ \text{add 1 and 1};\ 2 \\ \underline{58} & \text{add } 5, 8 \quad\ 13;\ \text{add 1 and 3};\ 4 \end{array}\right\} 4 \times 2 = 8 \\
2032 \\
\underline{1273} \\
15\,762\ \text{add } 1, 5, 7, 6, 2;\ 21;\ \text{add 2 and 1};\ 3, \text{ hence answer is incorrect.} \\
\qquad\quad \text{add } 1, 4, 7, 3, 2;\ 17;\ \text{add 1 and 7};\ 8 \text{ hence 14732 is correct.}
\end{array}$$

For criticisms see *d* under addition.

Checks for division:

a. Multiply the divisor and the quotient to obtain dividend.

b. Divide the dividend by the quotient to obtain the divisor.

c. Casting out nines and reducing to a digit when there is no remainder in the answer. This is an interesting but unreliable procedure.

Short Cuts

Because teachers are not convinced of their worth, short cuts have not as yet attained their rightful place in arithmetic.

Too often they are regarded as numerical tricks lacking sound mathematical basis.

Value of short cuts. 1. *Economy of time.* The obvious value of a short cut lies in its economy of time and labor. The time and effort so saved may be used to solve more problems and thus inculcate greater arithmetic skill.

2. *Added interest in mechanical processes.* In the solution of problems interest usually attaches to planning the solution and obtaining the result, but never to the mechanical computations. Short cuts tend to make these very mechanical operations thought-provoking and therefore interesting.

In multiplying 5423 by 279, there is no interest in multiplying by 9, by 7, and by 2, and in adding the three partial products. Instead of following the long process teach children to multiply 5423 by 9, and then to multiply the partial product by 3, because 27 is 3 times 9, thus:

$$\begin{array}{r l} 5423 & \\ \underline{279} & \\ 48807 & \\ \underline{146421} & \text{obtained by multiplying 48807 by 3. } (27=9\times 3) \\ 1\,513\,017 & \end{array}$$

To discover that 3 times the partial product is the same as 27 times the multiplicand provides an application of the ratio idea of number that lends interest to this mechanical operation. This method applies to multipliers like 648, 546, 549, 729, 819, 459, 728, 497 . . . a series of great length.

3. *A deeper insight into number.* The preceding illustration suggests that the application of short cuts gives the pupil a deeper insight into number relations, that is, into ratios. The child who learns the short method of multiplying by 12½ (add 2 ciphers and divide by 8), and by 125 (add 3 ciphers and divide by 8), has learned some of the implications in the proportion

$$\tfrac{1}{8} : 1 = 12\tfrac{1}{2} : 100 = 125 : 1000.$$

Every short cut gives a deeper understanding of the relations among numbers and shows more clearly the significance of the ratio conception of number.

4. *Greater skill in manipulation.* A value of short cuts that comes as a sequel to the deeper appreciation of number relations is increased skill in number manipulation. The illustrations above show clearly that greater dexterity follows the understanding and the use of short cuts.

Cautions in introducing short cuts. 1. *Not prematurely.* Short cuts must be introduced only after the long process is thoroughly understood. To teach short cuts before the longer processes are understood reduces them to mere tricks of computation. A short cut must be rational and contribute to a deeper comprehension of the functions of number.

2. *Short cuts not as mere mnemonics.* It follows, therefore, that a short cut must not be taught without mathematical justification. The method of testing muliplication by reducing to a digit is an illustration of an arithmetical process founded on a mathematical principle that cannot be demonstrated to the pupils. So, too, multiplying 92 by 94 and obtaining the product by writing 48 to the right and 86 to the left, 8648, is another example of a short cut that should not be taught. The following is the procedure: 96×89; take 96 from 100 and 89 from 100, thus obtaining 4 and 11; 4×11=44; now take 4 from 89 or 11 from 96 and obtain 85; the answer is 8544. When the numbers are 88×85, the process becomes confused and the child is lost because the short form of multiplying 96 by 89 is based on a principle beyond him.

3. *Short cuts taught must be susceptible of general application.* The short form of multiplication by 15 (×10; ½ of this product; add the two numbers), or by 11 (54×11, write 5, write 4, insert their sum between the 5 and the 4, 594), or by 41, 21, 61 (multiply by 40, 20, 60 and add the multiplicand to the product), are illustrations of processes that are not worth the teaching. To multiply by 15 or by 11 or by 71 directly will in the end save time and make for greater accuracy. To multiply 67 by 11 by writing 6 and 7 and inserting their sum will give 6137, a product obviously incorrect; to

require children to learn that when the sum of the two digits is greater than 9, the 1 must be added to the tens' digit makes the short multiplication too cumbersome a process. A short cut that has exceptions should itself be excepted from the list of forms worth teaching.

4. *Short cuts worth teaching should be reduced to habit.* A short cut must not be taught as an end in itself. Too often the teacher assigns an example to the class and adds, "Be sure to use the short cut." Those who solve the example by the long process are usually marked partly or wholly wrong. But when the suggestion is not given to the class almost all the pupils follow the long process from force of habit. Unless they are used spontaneously by the children themselves, short cuts are useless.

To make the use of short cuts habitual, it is necessary to assign very few new ones to each grade. This gives the class opportunity for sufficient drill. To require a teacher to teach almost all short cuts is a practice which makes impossible the reduction of the abbreviated solutions to habit. Teachers must so frame their examples and problems that the numbers will encourage children to use the short cuts that were taught. Thus, instead of asking for the number of gallons of water lost in 254 days at an average daily leakage of 536 gallons, the former number may be changed to 246 or 486 or 497 so that the short cut may be employed. Most problems can be made to yield practice in the thought phase of arithmetic and in the use of short cuts. Unless the short cut is employed from force of habit, it becomes merely an additional burden.

PARTIAL LIST OF SHORT CUTS

Addition and Subtraction

1. To add 58 and 27; add 58 + 20 + 7; say 58, 78, 85.
2. To take 27 from 85 reverse the process; say 85, 65, 58.

Multiplication

1. To multiply by
    25, add 2 ciphers and divide by 4.

12½, add 2 ciphers and divide by 8.
250, add 3 ciphers and divide by 4.
125, add 3 ciphers and divide by 8.
16⅔, add 2 ciphers and divide by 6.

2. To multiply by

75, add 2 ciphers, divide by 4, multiply by 3.
375, add 3 ciphers, divide by 8, multiply by 3.
.625, add 3 ciphers, divide by 8, multiply by 5.
750, add 3 ciphers, divide by 4, multiply by 3.
83⅓, add 2 ciphers, divide by 6, multiply by 5.
62½, add 2 ciphers, divide by 8, multiply by 5.

This short cut is generally not worth using, as it may involve as much computation as the long process.

3. To multiply by 10, 100, 1000, add the required number of zeros.

4. To multiply by 729, 546, 549, etc., proceed as explained above.

5. To multiply by a number containing two factors.

$524 \times 64\ (8 \times 8) = 524 \times 8 \times 8$ or

$$\begin{array}{r} 524 \\ \times 8 \\ \hline 4192 \\ \times 8 \\ \hline 33\,536 \end{array}$$

$4632 \times 175 = 4632 \times 25 \times 7$

$$\begin{array}{r} 4\overline{)463200} \\ 115\,800 \times 7 = 810\,600. \end{array}$$

The long process involves two multiplications and one addition; the short process employs two successive multiplications and eliminates the addition of partial products. *This is of doubtful value.*

Division

1. To divide by 10, 100, 1000, move the decimal point the required number of places.

2. To divide by

25, move decimal point 2 places left and multiply by 4.
125, move decimal point 3 places left and multiply by 8.
250, move decimal point 3 places left and multiply by 4.

16⅔, move decimal point 2 places left and multiply by 6.
12½, move decimal point 2 places left and multiply by 8.

3. To divide by a number that is a multiple of two factors, divide by the two factors.

$33536 \div 64 (8 \times 8)$ $\qquad$ $33536 \div 8 = 4192$
$4192 \div 8 = 524$ *Ans.*

Fractions

1. To add fractions with numerators of 1: $\frac{1}{3} + \frac{1}{5} = \frac{3 + 5}{3 \times 5} = \frac{8}{15}$.
2. To add any two fractions, for example,

$$\frac{2}{3} + \frac{3}{4} = \frac{(2 \times 4) + (3 \times 3)}{3 \times 4}$$

Shortened Algorism

1. In multiplying or dividing numbers by 1¼, 1½, etc., it is not necessary to rewrite the numbers; the abbreviated algorism is preferable.

$$\begin{array}{r} 254 \times 1\frac{1}{4} \\ 63\frac{1}{2} \\ \hline 317\frac{1}{2} \end{array} \quad \text{rather than} \quad \begin{array}{r} 254 \\ 1\frac{1}{4} \\ \hline 254 \\ 63\frac{1}{2} \\ \hline 317\frac{1}{2} \end{array}$$

2. When the cost is set at .50, .25, .75, .12½, it is always better to use the fractional part of a dollar, for example, ½, ¼, ¾, ⅛.
3. In taking a per cent of a number it is better to use the fractional equivalent in all cases where it is easily recognized. For example, 37½%, 83⅓%, 75%, and 12½%.
4. In examples where the successive steps involve a series of multiplications and divisions the cancellation form is preferable to the performance of each process separately.

In the problem, "Find the cost of filling a tank measuring 4 × 5 × 3 ft. with gasoline at 21 cents a gallon," there are two procedures:

a. The long way

$4 \times 5 \times 3 = 60$ cu. ft. in the tank.
$60 \times 1728 = 103{,}680$, cu. in. in the tank.

$$
\begin{array}{r@{\,}l}
 & 448.83\frac{9}{77}, \text{ gal. in the tank} \\
231) & \overline{103680.00} \\
 & \underline{924} \\
 & 1128 \\
 & \underline{924} \\
 & 2040 \\
 & \underline{1848} \\
 & 1920 \\
 & \underline{1848} \\
 & 720 \\
 & \underline{693} \\
 & 27
\end{array}
\qquad \frac{27}{231} = \frac{9}{77}
$$

$$
448.83\tfrac{9}{77} \times \$.21 = \begin{array}{r}
448.83\frac{9}{77} \\
\underline{.21} \\
448\ 83 \\
8976\ 6 \\
\underline{2\frac{5}{11}} \\
\$9425\ 45\frac{5}{11}
\end{array} = \$94.25, \text{ cost.}
\qquad \frac{9}{77} \times 21 = \frac{189}{77} = 2\frac{35}{77} = 2\frac{5}{11}
$$

b. The short form

$$
\frac{4 \times 5 \times \overset{1}{\cancel{9}} \times 1728 \times \overset{.03}{\cancel{.21}}}{\underset{\underset{11}{77}}{\cancel{231}}} = \frac{60 \times 1728}{11} = \frac{1036.80}{11} = \$94.25\tfrac{5}{11} = \$94.25, \text{ cost.}
$$

Application of Ratio Idea.

A ship travels 190 mi. in 6 hrs.; what distance will be covered in 18 hrs.?

a. $\frac{1}{6}$ of 190 = $31\frac{2}{3}$ mi. per hr. $31\frac{2}{3} \times 18$ = distance in 18 hrs.

b. Ratio method: 18 = 3 × 6, hence 190 mi. × 3 = 570 mi. Any illustration in the discussion of the ratio method of solving problems will show the economy of using this process, when practical, in contrast to the method of unitary analysis.

CHAPTER XXV

TEACHING COMMON FRACTIONS

Prevailing ignorance of fractions. Carefully planned and administered tests in arithmetic reveal a surprising ignorance among adolescents of the basic nature of fractions. An incredibly large part of seventh- and eighth-grade children who are tested do not know that numerator and denominator may be multiplied and divided by the same number; that numerator and denominator may not be increased or decreased through the addition or subtraction of the same number; that numerator and denominator have very different functions; that increasing the denominator decreases the value of the fraction; that in $\frac{2}{3}$ of $\frac{5}{7}$ we multiply the numerators for the numerator in the answer and the denominators for the denominator in the answer; that $\frac{2}{3}$ of $\frac{5}{7}$ and $\frac{2}{3} \times \frac{5}{7}$ are arithmetically identical; that $\frac{2}{3}+\frac{4}{5}$ is not the same as $\frac{2+4}{3+5}$ or $\frac{6}{8}$; that $\frac{2}{3}$ means 2:3; that $\frac{2}{3}$ means $2 \div 3$; that $\frac{4}{4}$ and $\frac{6}{6}$ have the same value. It is estimated that about half of an unselected group of children of seventh and eighth grades do not understand the fundamental functions of fractions.[1]

Among the most important causes of this deplorable ignorance we may mention: too little objective teaching, too few purposeful activities, too early teaching of fractions, ungraded subject-matter, overinsistence on definition, too little oral work, infrequent diagnostic testing and inadequate individual practice material. These are the weaknesses in teaching against which teachers must always be on guard.

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[1] Raleigh Schorling, "A Tentative List of Objectives in the Teaching of Junior-High-School Mathematics" (Ann Arbor, Mich., Geo. Wahr, 1925), pp. 23–24. Quoted by R. L. Morton in *Teaching Arithmetic in the Intermediate Grades* (Silver Burdett & Co.), p. 109.

Common or decimal fractions first? Mathematically considered decimal fractions ought to precede common or vulgar[2] fractions because the former are more closely related to the system of notation in whole numbers. For psychological reasons we teach common fractions first. The decimal fraction has a denominator too large for convenient objective demonstrations. In the life of the child the fractions ½ and ¼ arise naturally and frequently; ⅒, rather infrequently. For these reasons decimal fractions are taught after common fractions are understood and mastered.

Incidental learning of fractions. Because most courses of study begin systematic work with fractions in the fourth year, teachers must not assume that common fractions should not be introduced much earlier. In the first or second grade children learn to identify ½, ¼, and ¾ by folding paper, by cutting objects or dividing quantities into parts, by making purchases for a half, or a quarter of a dollar—in a word, by living in a modern community. Children can no more avoid certain common fractions than they can avoid the ordinary range of whole numbers. To be sure, they say that *half* means only a "part" or a "piece," but they soon see some of its implications. Two six-year-old children dividing a bar of chocolate insist on a true half of the whole for each.

Gradually, through manual experiences and directed activities, children learn ¾ of a string, ½ of 6 pencils, ¼ of a line, and the like. Through such concrete experiences as grouping fellow pupils, apportioning beads or marbles, folding paper, using money, and distributing class supplies children learn $\frac{1}{2}+\frac{1}{2}=1$; $\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1$; $\frac{3}{4}-\frac{1}{4}=\frac{2}{4}$; $\frac{2}{4}=\frac{1}{2}$; $\frac{1}{4}\times 3=\frac{3}{4}$; $1\div\frac{1}{4}=4$. They are now ready to work from large charts drawn in contrasting colors to set off the equivalent parts. The following charts, or sheets of paper 8×4 inches, suitably folded by each child, can be used to find the answers to the examples listed below:

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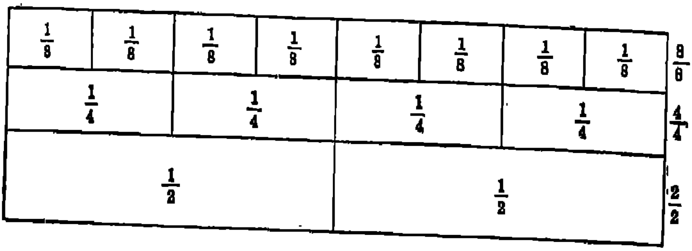
[2] *Vulgar* is the old adjective meaning, of course, not "coarse" but "common." In an older day a commoner was referred to as a vulgar person.

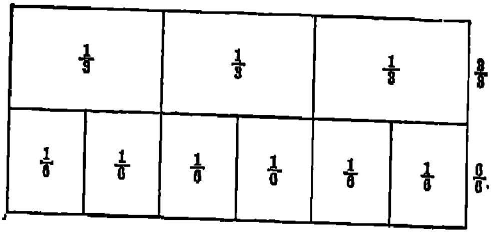


$\frac{1}{4}+\frac{1}{2}$
$\frac{1}{8}+\frac{3}{8}$
$\frac{1}{4}+\frac{1}{8}$
$\frac{1}{4}+\frac{3}{8}$
$\frac{3}{4}+\frac{1}{8}$

$\frac{7}{8}-\frac{1}{8}$
$\frac{3}{4}-\frac{1}{4}$
$\frac{1}{2}-\frac{1}{4}$
$\frac{1}{4}-\frac{1}{8}$
$\frac{7}{8}-\frac{1}{4}$

$\frac{1}{8}\times 3$
$\frac{1}{2}\times 2$
$\frac{1}{4}\times 3$
$\frac{3}{8}\times 2$
$\frac{1}{4}\times 2$

$\frac{3}{8}\div 3$
$\frac{3}{4}\div 3$
$\frac{5}{8}\div 5$
$\frac{6}{8}\div 3$
$\frac{1}{2}\div 2$



By manipulating a group of objects children can be led to find ½ of 4, ½ of 8, ¼ of 8 and ¼ of 4. Problems in finding what fractional part one number is of another can also be given because they can be made concrete:

Had 4¢, spent ½ of it; what did I spend?
Had 4¢, spent 2¢, what part of my money did I spend?

Had 8¢, spent ½ of it, what did I spend?
Had 8¢, spent 4¢, what part of my money did I spend?

After the arithmetical situations are understood the first of each pair of the above questions may be omitted and the children may then be asked, "Had 6 marbles and lost 3; what part of my marbles did I lose?"

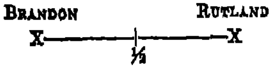
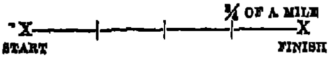
All of this work in fractions is almost incidental and free from rationalization, definitions, and rules. It is always concrete and often is written in the form, "1 quarter+2 quarters are 3 quarters," until the children learn to focus attention on the numerator. In the first three years the aim is to teach the child those computations with fractions that the untaught person learns from actual experience.

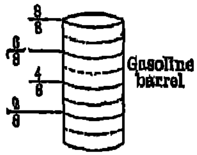
Systematic instruction in fractions: Graded steps in the development of fractions. Through concrete experiences and purposeful activities children must learn the following progressive ideas of fractions:

1. ½, ¼, ⅛ of a single object, as ¼ of this sheet of paper.
2. ½, ¼, ⅛ of a group of objects, as ¼ of these 8 sheets of papers.
3. ½, ¼, ⅛ of a definite unit, as ¼ of a foot is how many inches?
4. ½, ¼, ⅛ of a group of definite and standardized units, as ¼ of $8; ½ of 50¢.
5. ¾, ⅝, ⅔ of a single object or standardized unit, as ¾ of $1; ⅝ of 1 lb.
6. ¾, ⅝, ⅔ of a group of standardized units, as ¾ of $4; ⅝ of 24 lb.
7. $\frac{2}{2}$, $\frac{4}{4}$, $\frac{8}{8}$ of an object or standardized unit refers to the thing itself.
8. $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{8}{8}$ of a group of objects or units refers to the whole group.
9. Improper fractions as $\frac{5}{4}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{9}{8}$ of a group of objects or units.
10. The mixed numbers as $1\frac{1}{4}=\frac{5}{4}$, $1\frac{1}{2}=\frac{3}{2}$, $1\frac{1}{3}=\frac{4}{3}$, $1\frac{1}{8}=\frac{9}{8}$.

The four ideas in the concept of fraction. After an adequate groundwork has been laid for work in fractions four main ideas should be taught as the constituents of the rich concept known as *fraction.* These are (1) a fraction is one or more of the equal parts of *a unit;* (2) a fraction is *one* of the equal parts of *a number of units;* (3) a fraction is an indicated division; (4) a fraction is an expressed ratio.

A fraction as one or more of the equal parts of a unit. The children have had enough experiences with fractions to answer the teacher's questions very readily. Reference to concrete material helps the children to include the idea of equality in their statements.

Teacher Asks	*Teacher Elicits from the Children*
1. I need ½ of this paper. How will you give it to me?	1. I will fold the paper into two equal parts and give you one.
2. A man traveled ½ the distance between Brandon and Rutland. What does ½ tell you? Teacher draws on blackboard:  	2. The distance between Brandon and Rutland is divided into two equal parts and the man traveled one of them.
3. Bill ran ¾ of a mile. What does ¾ tell you? Teacher draws on blackboard:  	3. A mile was divided into four equal parts. Bill ran three of them.
4. The man in the service station noticed that he sold ⅝ of his gasoline. What does ⅝ tell you? Teacher draws on blackboard:	4. The barrel is divided into eight equal parts and five of them were sold.



These fractions are now written in a horizontal line, thus ½, ¾, ⅝, as the teacher asks, "How did you know into how many equal parts to divide the paper, the distance, the mile, the tank? How did you know how many of the equal parts to take?" The blackboard information now takes on a new form:

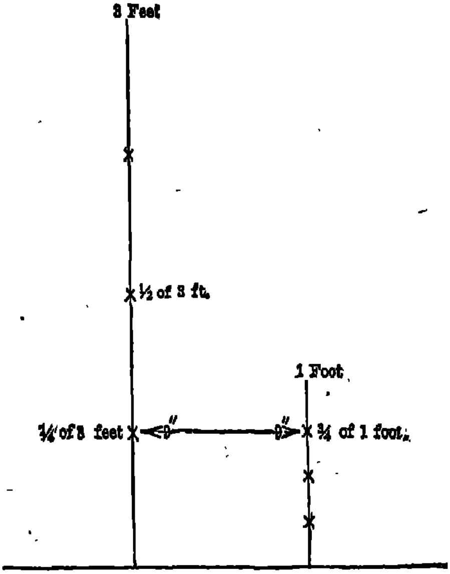
$$\frac{1}{2},\ \frac{3}{4},\ \frac{5}{8}\quad \frac{\text{tell us how many of the equal parts to take}}{\text{tell us into how many equal parts to divide the thing}}$$

We must note that the teacher writes only what the children say. The class is now ready to learn three terms: *numerator, denominator,* and *fraction.* The teacher explains that the words *numerator* and *denominator* are used for the long expressions above and below the line. Adding these two terms to the blackboard material, we have:

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{\text{tell us how many of the equal parts to take}}{\text{tell us into how many equal parts to divide the thing}}, \frac{\text{Numerator}}{\text{Denominator}}$$

The lesson is capped when the definition of a fraction, "one or more equal parts of a thing" is elicited.

A fraction as one of the equal parts of a number of units. The full understanding of a fraction requires a knowledge of the fact that ¾ may be regarded as ¾ of 1 and ¼ of 3; that ⅖ may be interpreted as ⅖ of 1 or ⅕ of 2. To make this second relationship clear, children are asked to tell the difference between ¾ of 1 ft. and ¼ of 3 feet. Naturally some think ¾ of 1 is more than ¼ of 3. Others disagree. The teacher then suggests that dispute will prove nothing, but that a concrete demonstration may show the correct answer. To illustrate the interpretations two vertical lines are drawn on



the board, one, 12 inches, the other 36. Measurements are made, and ¾ of 1 foot is found to be 9 inches, while ¼ of 3 feet is also 9 inches. Similar demonstrations can be made by folding paper or by using any of the tables of weights and measures taught in these grades. Children should be required to tell the two stories of ¾, of ⅔, of ⅝, of ⅖, thus ¾ means ¾ of 1 and also ¼ of 3. A two-minute exercise like the following is helpful and not unpopular with children.

Write the other form of each of the following:

⅓ of 2 = ___ of ___.
¾ of 1 = ___ of ___.
⅓ of 7 = ___ of ___.
⅚ of 1 = ___ of ___.
⅜ of 1 = ___ of ___.
⅛ of 5 = ___ of ___.

It is futile to argue that in no business situation is one required to know that ⅔ of 1 = ⅓ of 2, because all will admit the fact. These ideas that make up the concept *fraction* are necessary for that adequate understanding of number relations which insures intelligent and accurate number manipulations in any common industrial or commercial situation that may arise later. To prepare for the practical demands of business one must learn much that is foundational in the structure of arithmetical ideas and skills.

A fraction as an indicated division. That a fraction is an indicated division must be indelibly impressed on each child. Much of the later work—not only in common fractions but in decimal fractions and in cancellation—depends on a clear comprehension of this basic fact. Children can be led to discover this relationship by listing on the board fractions and their interpretation as part of a unit or of a group, as shown under *A* below. The teacher then asks, "How can we obtain ¼ of 3?" and elicits, "Divide 3 into 4 equal parts." The teacher writes what the children say and obtains the material shown under *B*. Section *C* is shown as an abbreviated form of *B*.

A	*B*	*C*
½ = ½ of 1,	that is, 1 is divided into 2 equal parts,	1 ÷ 2
¼ = ¼ of 1,	that is, 1 is divided into 4 equal parts,	1 ÷ 4

$\frac{1}{5} = \frac{1}{5}$ of 1, that is, 1 is divided into 5 equal parts, $1 \div 5$

$\frac{3}{4} = \frac{1}{4}$ of 3, that is, 3 are divided into 4 equal parts, $3 \div 4$

$\frac{4}{5} = \frac{1}{5}$ of 4, that is, 4 are divided into 5 equal parts, $4 \div 5$

$\frac{5}{6} = \frac{1}{6}$ of 5, that is, 5 are divided into 6 equal parts, $5 \div 6$

The children are then led to recognize that 1, 1, 1, 3, 4, 5, are the numerators in each case and that 2, 4, 5, 4, 5, 6, are the denominators. The following tabulation readily leads to the generalization that a fraction is the equivalent of its numerator divided by its denominator.

$$\frac{1}{2} = 1 \div 2 = N \div D$$
$$\frac{3}{4} = 3 \div 4 = N \div D$$
$$\frac{4}{5} = 4 \div 5 = N \div D$$
$$\frac{5}{6} = 5 \div 6 = N \div D$$

Practice must now be given in telling the three stories of a fraction, thus $\frac{3}{4} =$ (*a*) $\frac{1}{4}$ of 3; (*b*) $\frac{3}{4}$ of 1; (*c*) $3 \div 4$. Exercises like the following help to fix these forms:

$2 \div 3 =$ ___.

$2 \div 3 = \frac{1}{3}$ of ___.

$2 \div 3 = \frac{2}{3}$ of ___.

$\frac{3}{8} =$ ___ $\div$ ___.

$\frac{3}{8} = \frac{1}{8}$ of ___.

$\frac{3}{8} =$ ___ of 1.

Once more we justify the teaching of a mathematical principle, not because of specific business needs, but rather because we are preparing young people for any reasonable arithmetical demand that business may make upon them.

A fraction as a ratio. The idea that a fraction expresses a ratio should not be taught in any given lesson, but should be developed gradually through cumulative experiences with fractions, especially in finding what fractional part one quantity is of another. The child who has 2 agates and loses 1 realizes that 1 of his 2 agates is $\frac{1}{2}$ of his possession. As a result of extensive experience and maturing insight, $\frac{1}{2}$ assumes a new significance; 1 out of 2, 1 as compared with 2, 1 is to 2, 1:2. It seems reasonable to conclude that integers developed from fractional ideas rather than *vice versa;* fractions are more closely related to measurement of small quantities than integers. The primitive man who caught one deer while his neighbor brought in two, perceived, although he did not conceive the fact, that he had only one-half of his neighbor's quarry. In the measurement process we perceive the parts before we have estimated the whole; each part is a fraction of the whole. The ratio idea also grows with the solution of such examples as "A has $4 and needs $5; what part of the amount he needs has he now?" The answer ⅘ means 4 out of 5; the relation of what he had to what he needs is 4 to 5, hence ⅘ is equivalent to 4:5. Explanation of these relationships do not help an immature pupil. Repeated experience with increasing maturity lead to a gradual understanding. The method of teaching the ratio idea is given in greater detail in an appropriate chapter later in this book.

Limiting the scope of fractions. There is a growing tendency to use only the fractions ½, ⅓, ⅔, ¼, ¾, ⅛, ⅜, ⅕, ⅖, and ⅘ and to eliminate from school work such fractions as ⅚, $\frac{7}{12}$ and $\frac{4}{9}$. Why ignore these fractional forms? They are not as common as the fractions in the first list, but they do occur—not only in everyday relations, but in working with the acceptable fractions. ½ and ⅓ do produce ⅚; the sum of ¼ and ⅓ is $\frac{7}{12}$. If we teach ½, ¼, and ⅓, we must also teach those fractions which are produced by adding or multiplying these. Stone, who objects to "uncommon" fractions, nevertheless says, "In every school course there should be some attention to problems that seldom occur in the mere 'social needs,' problems of a type never met before by the pupils,"[3] to develop greater power to see quantitative relations.

Reduction of fractions. *Technical meaning of the term "reduction."* Children who know the word *reduction* are often confused by the technical meaning of the term—not by the process of reducing a fraction. They use the word in the dictionary sense of making less or smaller. But when fractions are reduced they may take on a larger as well as a smaller form. The essential characteristic of *reduction,* in this technical sense, is that the form is changed but the value remains unaltered. In $\frac{2}{4}=\frac{1}{2}=\frac{4}{8}$ we have $\frac{2}{4}$ reduced to lower terms, $\frac{1}{2}$, and to higher terms, $\frac{4}{8}$. Both changes are known as reduction; neither change affects the value of $\frac{2}{4}$.

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[3] John C. Stone, *The Stone Arithmetic: The Teacher's Manual, Grades 1 to 6* (Benjamin H. Sanborn & Co., 1928), p. 19.

Kinds of reduction. Five useful kinds of reduction must be taught in the elementary grades:

1. Reduction to lowest terms, $\frac{4}{6}=\frac{2}{3}$; $\frac{8}{12}=\frac{2}{3}$; $\frac{4}{8}=\frac{1}{2}$.
2. Reduction to common denominators, $\frac{2}{3}$ and $\frac{5}{6}=\frac{4}{6}$ and $\frac{5}{6}$; $\frac{2}{3}$, $\frac{5}{6}$ and $\frac{1}{4}=\frac{8}{12}$, $\frac{10}{12}$ and $\frac{3}{12}$ respectively.
3. Reduction to higher denominators—changing quarters to eighths.
4. Reduction to improper fractions, $1\frac{1}{4}=\frac{5}{4}$; $2\frac{5}{8}=\frac{21}{8}$.
5. Reduction to mixed numbers, $\frac{5}{4}=1\frac{1}{4}$; $\frac{21}{8}=2\frac{5}{8}$.

Teaching reduction of fractions. The teaching problems in these five types of reduction are identical. In each we must be sure to incorporate the following teaching practices: (1) motivation; (2) the use of concrete materials; (3) guiding the class to realize the importance of dispensing with the objective aids; (4) helping the class to formulate the generalization; and (5) the application of the generalization by solving many examples. The sequence of these practices may be changed to suit circumstances, but in the final analysis each will be found in an effective lesson on reduction. For an illustration of this technique, we refer the reader to the lesson on the reduction of fractions to lowest terms as given in Chapter XV in which occurs the discussion of the inductive recitation. After each form of reduction is taught, cumulative drills like the following should be mimeographed and given to the children as useful practice material for spare minutes.

$1=\frac{2}{2}$ $\frac{2}{2}=1$ $1=\frac{3}{3}$ $\frac{3}{3}=1$ $\frac{1}{2}$ and $\frac{1}{3}$ are the same as $\frac{?}{6}$ and $\frac{?}{6}$

$5=\frac{?}{2}$ $\frac{10}{2}=?$ $1\frac{1}{2}=\frac{?}{2}$ $\frac{3}{2}=1\frac{?}{2}$ $\frac{1}{4}$ and $\frac{1}{3}$ are the same as $\frac{?}{12}$ and $\frac{?}{12}$

$1=\frac{4}{4}$ $\frac{4}{4}=1$ $1=\frac{8}{8}$ $\frac{5}{2}=?\frac{?}{2}$ $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$ are the same as $\frac{?}{12}$, $\frac{?}{12}$, and $\frac{?}{12}$

$3=\frac{?}{4}$ $\frac{12}{4}=?$ $1\frac{2}{3}=\frac{?}{3}$ $\frac{9}{2}=?\frac{?}{2}$ $\frac{1}{3}=\frac{2}{6}$, $\frac{2}{3}=\frac{?}{6}$; $\frac{1}{4}=\frac{2}{8}$, $\frac{3}{4}=\frac{?}{8}$

Reducing to lowest terms by the greatest common divisor. In practical arithmetical operations there is almost no occasion for finding the greatest common divisor. Where fractions are to be reduced to lowest terms, the method of successive divisions is adequate; thus: $^{14}/_{42} = {^7}/_{21} = {^1}/_{3}$. Few fractions in business computations are as clumsy as $^{14}/_{42}$, and even with such a fraction the greatest common divisor is an arithmetical luxury.

Where the long Euclidean method must be taught because of course-of-study assignment, no attempt should be made at rationalization.

G. C. D. of 58 and 126

```
       2
  58)126
     116  5
     ------
      10)58
         50  1
         -----
          8)10
             8 4
             ---
             2)8
```

G. C. D. = 2.

To teach children to reduce fractions to lowest terms by resolving both numerator and denominator into their factors and then discovering the highest common factor also involves needless waste; this can be seen in the case cited below.

$$\frac{14}{42} = \frac{2 \times 7}{2 \times 7 \times 3}, \text{ hence the H. C. F. is } 2 \times 7, \text{ hence}$$

$$\begin{array}{r|l} \div 14 & \dfrac{14}{42} = \dfrac{1}{3} \text{ or } \dfrac{14 \div 14}{42 \div 14} = \dfrac{1}{3} \\ \div 14 & \end{array}$$

Here, again, too much time is consumed and the method is unnecessary for the type of fractions generally found in commercial and industrial life.

In every instance the reader will find that the method of trial (guess) and correction is superior to systematic analysis into prime factors. A few suggestions for testing divisibility of numbers may be helpful, but these should be pointed out *incidentally* and never committed to memory:

1. A number is divisible by 2 if the last digit is even.
2. A number is divisible by 3 if the sum of the digits is divisible by 3.
3. A number is divisible by 4 if the last two digits are divisible by 4.
4. A number is divisible by 5 if the last digit is 0 or 5.
5. A number is divisible by 6 if it is divisible by 2 and 3 or if it is even and rule No. 2 applies.
6. A number is divisible by 8 if the last 3 digits are divisible by 8.
7. A number is divisible by 9 if the sum of the digits is divisible by 9.

Multiples and factors. *Their importance.* A knowledge of multiples is necessary in all reductions of fractions, in addition and subtraction of fractions with dissimilar denominators, in cancellation, in proportion, in ratio solutions, in short cuts, and, indirectly, in most processes in arithmetic. Multiples imply factors, for the two classes of numbers are intimately related. To know that 56 is a multiple of 8 and 7 or that 8 and 7 are factors of 56 enables one to solve division examples by short cuts—that is, by two successive divisions.

Teaching multiples and factors. To develop the concepts *multiple* and *factor* set the stage for a number game or a number puzzle. Write small numbers in a column, and ask the class to give the larger numbers that will be divided exactly by these smaller numbers, thus:

Teacher gives:	*Children supply:*
2, 3,	6, 12, 18, 24, etc.
4, 5,	20, 40, 60, 80, 100, etc.
8, 6,	24, 48, etc.
3, 5, 6,	30, 60, etc.
2, 5, 4,	20, 40, 60, etc.
2, 3, 5,	30, 60, 90, etc.

The winners of the game are those who can give the large numbers most readily or who can give most of the numbers. The teacher then elicits the relationship existing between numbers in the first column and those in the second. When the children formulate the function of each the teacher writes it in the most suitable and economical language. Under the two columns of figures, we now see on the blackboard the following:

Can divide the larger number without a remainder.	Can contain or can be divided by the smaller numbers, without a remainder.
Factors	*Multiples*

The words *factors* and *multiples* are now introduced in the appropriate columns as shown above. Children (1) see the function, (2) formulate it, and (3) learn the technical term for it. The expression *least common multiple* is introduced as the best descriptive name for the first or smallest multiple in each line.

Finding L. C. M. or L. C. D. by inspection. Much valuable time is often lost because teachers interpret the expression *by inspection* to mean *without any computation whatever,* or *guessing.* In seeking the least common denominator for the fractions $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{4}{9}$, children may be seen gazing aimlessly at 4, 6, and 9, hoping to attain by mere chance the least common multiple. They usually select, in a casual way, an unsuitable number. *By inspection* must be interpreted to mean *without written computation;* hence children must be taught a simple means of determining the least common denominator. The following are some of the devices suggested—the first will be found the most satisfactory:

1. Select the highest factor and try successive multiples until the necessary number is found.

   4, 6, 9; 9 is selected; 9×1, 9×2, 9×3, $9 \times 4 = 36$. L. C. D., 36

2. Take the first two numbers and find their L. C. M., then find the L. C. M. of that and the next number.

   4, 6, 9; L. C. M. of 4 and 6 is 12; L. C. M. of 12 and 9 is 36.

3. Disregard those smaller denominators that are factors of larger ones; then find the L. C. M. of the remaining denominators.

    2, 3, 4, 8, 6; disregard 2 and 4 because of 8; disregard 3 because of 6; find the L. C. M. of 8 and 6; 8 × 1; 8 × 3; 8 × 3 = 24. L. C. D. is 24.

4. Write out the successive multiples of each denominator, then select the first multiple that is common to all.

    4, 8, 12, 16, 20, 24, 28, 32, 36, 40
    6, 12, 18, 24, 30, 36, 42, 48, 54, 60
    9, 18, 27, 36 hence 36

This method often involves the writing of more multiples than are necessary and may, therefore, be longer, but it shows clearly the meaning of *least* and *common.*

Thorndike is very insistent on the elimination of the least common denominator. Teach children to think of given fractions as fourths, sixths, ninths, twelfths, fifteenths, eighteenths, or twenty-fourths, as the case may require. To add $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{5}{6}$ children should think of each addend as twelfths and obtain $\frac{8}{12}$, $\frac{3}{12}$, and $\frac{10}{12}$, respectively. The difference between the common practice of teaching children to find and use the least common denominator and the practice advocated by Thorndike, resolves itself into teaching a term or not teaching it. This is really a quarrel over a word. A common technical term is a convenience once it is thoroughly understood: teacher and class can then identify a necessary element with a minimum of language.

To aid the children to reduce fractions to equivalent fractions that have the same least denominator, the chart on page 348, hung in front of the class, is very serviceable. If the fractions are $\frac{2}{3}$ and $\frac{3}{4}$ the child looks for the first column that contains the same denominator for thirds and fourths and finds twelfths; if the fractions are $\frac{5}{6}$ and $\frac{3}{4}$ the child finds that the first suitable column is, again, the one marked twelfths.

L. C. D. not found by inspection. When many fractions having large denominators must be added, the least common

Fractions	The Common Denominators											
1/2	3/6	4/8		5/10	6/12	7/14		8/16	9/18	10/20		12/24
1/3	2/6		3/9		4/12		5/15		6/18		7/21	8/24
1/4		2/8			3/12			4/16		5/20		6/24
1/5				2/10			3/15			4/20		
1/6	1/6				2/12				3/18			4/24
1/7						2/14					3/21	
1/8		1/8						2/16				3/24
1/9			1/9						2/18			
1/10				1/10						2/20		
1/12					1/12							2/24

denominator cannot be found by inspection. The long process of finding the least common multiple must then be used. It was suggested in outlining the scope of elementary-school arithmetic that such fractions should be excluded, but many courses of study demand these and children must therefore be taught how to add 5/12, 9/14, and 7/18. This long process of finding the L. C. D. should be presented in a deductive lesson with rigid exclusion of all rationalization. The teacher must show the process a number of times so that the class will learn through imitation. Any attempt at rationalization will become an additional source of confusion and will interfere with the early mastery of the process.

What should reduction of fractions give the child? As a result of a well developed program of class work in fractions and their reductions, children should show thorough mastery of "the six rules of fractions":

1. The effect of dividing numerator and denominator by the same number.
2. The effect of multiplying numerator and denominator by the same number.

3. The effect of multiplying the numerator alone.
4. The effect of dividing the numerator alone.
5. The effect of multiplying the denominator alone.
6. The effect of dividing the denominator alone.

These six rules should not be reduced to stereotyped sentences nor should they be taught seriatim as here listed. They should be learned incidentally in working with fractions. Periodic tests composed of questions like the following will give adequate emphasis to the six rules of fractions:

Which is larger ¼ or ⅛? ⅙ or ¼? ⅚ or ⅝? 7⁄12 or ⅞?
Which is smaller ⅛ or ½? ⅓ or ¼? ⅗ or ⅜?
Which is worth more ⅔ or ⅘? 1 or ⅚? 5⁄5 or 6⁄6? 8⁄8 or 1?
What does the multiplier 2 do to each of the following fractions?

$$\frac{3}{4\times2};\ \frac{3\times2}{4};\ \frac{3\times2}{4\times2};\ \frac{2}{2}\times2;\ \frac{2}{2\times2};\ \frac{1}{6}\times2;\ \frac{1}{6\times2};\ \frac{6\times2}{6\times2};\ \frac{1\times2}{6\times2}$$

A pint is what part of a quart? A quart is what part of a gallon?
A pint is what part of a gallon? 3 pints are what part of a gallon?
A cent is what part of a nickel? of a dime? A nickel is what part of a dime? Three cents are what part of a nickel? Three cents are what part of a dime?
What is 1⁄10 of a dollar? ½ of a dime? ⅕ of a nickel?
How many cents in ¾ of a dollar? 1⁄10 of a dollar? 5⁄10 of a dollar?

In the comparison of fractions children learn the significance of increasing and decreasing the size of each and of both terms of a fraction. They acquire a number sense that may save them from absurd statements like the following, so frequently made by eighth-grade children: $\frac{2}{3}+\frac{3}{4}=\frac{5}{7}$ or $\frac{6}{12}$; $\frac{2}{3}\times\frac{3}{4}=\frac{6}{7}$; $4\div\frac{1}{2}=2$; $4\times\frac{1}{2}=8$.

PROBLEMS FOR STUDY AND DISCUSSION

See end of Chapter XXVI

CHAPTER XXVI

TEACHING FUNDAMENTAL OPERATIONS WITH COMMON FRACTIONS

Addition and Subtraction of Fractions

Gradation in addition of fractions. Most of the teaching problems in addition and subtraction of fractions are greatly simplified by a nice gradation. We must therefore identify the successive difficulties, one at a time.

The Steps	*The Reason for the Gradation*
1. $\frac{1}{3}+\frac{1}{3}$; $\frac{1}{4}+\frac{2}{4}$; $\frac{1}{8}+\frac{3}{8}+\frac{1}{8}$.	1. Similar denominators, two or three addends, the sum less than 1.
2. $\frac{2}{5}+\frac{3}{5}$; $\frac{1}{4}+\frac{3}{4}$; $\frac{1}{4}+\frac{2}{4}+\frac{1}{4}$; $\frac{5}{8}+\frac{7}{8}$.	2. Similar denominators, two or three addends, the sum equal to 1 or more than 1.
3. $\frac{1}{2}+\frac{1}{4}$; $\frac{1}{8}+\frac{1}{4}$; $\frac{1}{3}+\frac{1}{6}$. With chart or diagram similar to those on page 336.	3. Dissimilar denominators, numerator always one, common denominator apparent from chart.
4. Same as 3. No chart.	4. Informal work in fractions and the practice with the chart provided in step 3 lead to mastery of the technique without the visual aid.
5. *a*. $\frac{3}{4}+\frac{1}{8}$; $\frac{2}{3}+\frac{1}{6}$ *b*. $\frac{3}{4}+\frac{5}{8}$; $\frac{2}{3}+\frac{5}{6}$	5. Dissimilar denominators but common denominator is apparent because it is in one of the addends. At first the sum is less than 1; later the sum is 1 or more than 1.

The Steps	*The Reason for the Gradation*
6. 5 + ⅔; 5 + 3⅔; ⅘ + 5; 5⅘ + 5.	6. One of the addends is a whole number; the other, a fraction or a mixed number.
7. 1⅚ + 2⅚	7. Both addends contain fractions whose sum is 1 or more than 1, thus necessitating reduction of the resulting improper fraction to a mixed number.
8. 2⅓ + 5⅙; 2⅔ + 5⅚	8. Same difficulties as in 5, 6, 7, with mixed numbers.
9. *a.* ⅓ + ¼; ½ + ⅓ *b.* ⅔ + ¾; ½ + ⅔ + ¼	9. Dissimilar denominators but the common denominator can be found by inspection even though it is in none of the addends. Thus, the L. C. D. is 12 in ⅓ and ¼ and 6 in ½ and ⅓, but none of the denominators is 12 or 6. In *a* the sum is less than 1; in *b* the sum is 1 or more than 1.
10. *a.* 2⅓ + 3¼; 1½ + 4⅓; *b.* 2⅔ + 3¾; 1½ + 2⅔ + 4¼	10. Same difficulties as in 9; all addends are mixed numbers. The whole numbers must be augmented, therefore, by the whole number resulting from the addition of the fractions.

The gradation is continued by increasing the addends to three, then four or more fractions or mixed numbers. It is well to intermingle addends that are whole numbers with addends that are either fractions or mixed numbers. A finer gradation is possible but hardly necessary. Fewer steps may be required in teaching bright pupils, especially under a system of homogeneous class organization.

Gradation in subtraction of fractions: We must now analyze the graded difficulties which arise in teaching subtraction of fractions.

The Steps	*The Reason for the Gradation*
1. $\frac{5}{8}-\frac{3}{8}$	1. Similar denominators.
2. $2\frac{5}{8}-\frac{3}{8}$; $5\frac{5}{8}-2\frac{3}{8}$	2. The conditions in step 1 are applied to mixed numbers.
3. $\frac{5}{8}-\frac{1}{4}$; $\frac{5}{6}-\frac{2}{3}$. With the aid of a chart if necessary.	3. Dissimilar denominators but the common denominator is either in the minuend or in the subtrahend.
4. $5\frac{5}{8}-2\frac{1}{4}$; $8\frac{5}{6}-3\frac{2}{3}$	4. Step 3 is now applied to mixed numbers.
5. *a*. $\frac{3}{4}-\frac{1}{3}$; $\frac{1}{2}-\frac{1}{3}$ *b*. $2\frac{3}{4}-1\frac{1}{3}$; $5\frac{1}{2}-\frac{1}{3}$	5. Dissimilar denominators but the common denominator is not to be seen in either minuend or subtrahend. In *b* there is really no new difficulty; the appearance is changed by introducing mixed numbers.
6. *a*. $3-\frac{1}{4}$; $6-\frac{4}{5}$ *b*. $3-1\frac{1}{4}$; $6-2\frac{4}{5}$	6. The problem of borrowing is introduced but the difficulty is minimized by making the minuend a whole number. $3-\frac{1}{4}$; $\frac{1}{4}$ from $\frac{4}{4}$; the $\frac{4}{4}$ need not be added to another fraction. In *b* the subtrahend is made a mixed number, but the entire situation is really the same.
7. $2\frac{1}{8}-\frac{5}{8}$; $7\frac{1}{6}-2\frac{5}{6}$	7. Borrowing is now carried one step further. When $\frac{8}{8}$ is obtained, we must add $\frac{1}{8}$ to produce $\frac{9}{8}$. Now we are ready to subtract the $\frac{5}{8}$ from the $\frac{9}{8}$. Similarly, $\frac{6}{6}$ is obtained, then augmented to $\frac{7}{6}$ before $\frac{5}{6}$ is subtracted. Note, however, that the process of reducing to a common denominator is not introduced in this step.

The Steps	*The Reason for the Gradation*
8. *a.* 4⅛ — ¾; 4⅛ — 2¾ *b.* 4½ — 1⅓; 5⅜ — 2⅔	8. Now the process of borrowing is made more difficult by the fact that the denominators in minuend and subtrahend must be reduced to a common denominator. Note that at first the denominator in either the minuend or the subtrahend is the common denominator. Later, in *b*, the common denominator must be found by inspection.

We must repeat that a greater number of graded steps can be formulated if necessary. For all practical teaching situations no finer gradation is necessary. Gifted children need not be led so slowly.

The algorism in addition. Two forms are generally used to-day. The reader is invited to analyze each carefully.

Form A

$$\begin{array}{r} \frac{1}{4}=\frac{3}{12} \\ \frac{1}{3}=\frac{4}{12} \\ \frac{5}{6}=\frac{10}{12} \\ \hline \frac{17}{12} \\ =1\frac{5}{12} \end{array} \qquad \begin{array}{r|r} 5 & \frac{1}{4}=\frac{3}{12} \\ 2 & \frac{1}{3}=\frac{4}{12} \\ 4 & \frac{5}{6}=\frac{10}{12} \\ \hline 11 & \frac{17}{12} \\ \multicolumn{1}{r}{} & =1\frac{5}{12} \end{array}$$

$$11+1\frac{5}{12}=12\frac{5}{12}$$

Form B

$$\begin{array}{r|l} \multicolumn{1}{r}{} & \quad 12 \\ \hline \frac{1}{4} & 3 \\ \frac{1}{3} & 4 \\ \frac{5}{6} & 10 \\ \hline \multicolumn{2}{r}{\frac{17}{12}=1\frac{5}{12}} \end{array} \qquad \begin{array}{r|l} \multicolumn{1}{r}{} & \quad 12 \\ \hline 5\frac{1}{4} & 3 \\ 2\frac{1}{3} & 4 \\ 4\frac{5}{6} & 10 \\ \hline \multicolumn{2}{l}{11 \quad \frac{17}{12}=1\frac{5}{12}} \end{array}$$

$$11+1\frac{5}{12}=12\frac{5}{12}$$

Form A shows the equivalent values of fractions and is therefore approved by many teachers. Others object to this algorism because children often add one of the common denominators with the numerators.

Form B is approved by some teachers because penmanship is reduced. But others point out that after children have added the numerator they forget *to write the total numerator above* the common denominator and give 17 as the sum of the three fractions.

It is obvious that no algorism is perfect. The important conclusion is that a uniform arrangement should be followed throughout a school system so that children will not be compelled to use first the one algorism and then the other as they go from one class to another.

The algorism in subtraction of fractions. In those instances when the fraction in the subtrahend is smaller than the fraction in the minuend the algorism of addition should be followed. When "borrowing" takes place a preliminary step helps to make the situation clear. This is illustrated in the following examples.

$7-2\frac{5}{8}$	$7\frac{1}{8}-2\frac{5}{8}$
$\begin{array}{r} 7=6\frac{8}{8} \\ 2\frac{5}{8}=2\frac{5}{8} \\ \hline 4\frac{3}{8} \end{array}$	$\begin{array}{r} 7\frac{1}{8}=6\frac{9}{8} \\ 2\frac{5}{8}=2\frac{5}{8} \\ \hline 4\frac{4}{8}=4\frac{1}{2} \end{array}$
$7\frac{1}{8}-2\frac{3}{4}$	$7\frac{1}{3}-2\frac{1}{2}$
$\begin{array}{r} 7\frac{1}{8}=6\frac{9}{8} \\ 2\frac{3}{4}=2\frac{6}{8} \\ \hline 4\frac{3}{8} \end{array}$	$\begin{array}{r} 7\frac{1}{3}=7\frac{2}{6}=6\frac{8}{6} \\ 2\frac{1}{2}=2\frac{3}{6} \\ \hline 4\frac{5}{6} \end{array}$

It is very important to retain the system of subtraction that is used with whole numbers. If the decomposition or reduction to equivalents was the method of subtraction taught then the procedure set forth above should be followed. Where Austrian or additive subtraction is used the additive subtraction should be applied to subtraction of fractions. This is illustrated on the left.

$$9-4\tfrac{1}{8}$$

$$\left.\begin{array}{l}9\\4\frac{1}{8}\end{array}\right\}\text{ the child must }\textit{think}\text{ of this as }\left\{\begin{array}{l}9\frac{8}{8}\text{ Child }\textit{thinks}\text{: }\frac{1}{8}\text{ and }\frac{7}{8}=\frac{8}{8};\text{ he writes: }\frac{7}{8}\\5\frac{1}{8}\\\hline 4\frac{7}{8}\text{ Child }\textit{thinks}\text{: }5\text{ and }4=9;\text{ he }\textit{writes}\text{: }4\end{array}\right.$$

$$9\tfrac{1}{4}-4\tfrac{3}{8}$$

$$\left.\begin{array}{l}9\frac{1}{4}\\4\frac{3}{8}\end{array}\right\}\text{ the child must }\textit{think}\text{ of this as }\left\{\begin{array}{l}9\frac{5}{4}\text{ This is now}\\\underline{5\frac{3}{8}}\text{ conceived as}\end{array}\right\}\begin{array}{l}9\frac{10}{8}\text{ Child }\textit{thinks}\text{: }\frac{3}{8}\text{ and }\frac{7}{8}=\frac{10}{8};\text{ he }\textit{writes}\text{: }\frac{7}{8}\\5\frac{3}{8}\text{ Child }\textit{thinks}\text{: }5\text{ and }4=9;\text{ he }\textit{writes}\text{: }4\\\hline 4\frac{7}{8}\end{array}$$

To those who subtract by a plan that involves no addition this looks very forbidding. This is to be expected because additive subtraction involves its own set of habits. Classroom experience seems to indicate that a third form of subtracting fractions gives results comparable to those obtained by the orthodox systems. In the example, $7\frac{3}{8}-2\frac{5}{6}$, we assume that there is no fractional part in the minuend and then add that fractional quantity to the tentative answer, thus,

$$7\frac{3}{8}-2\frac{5}{6}$$

$$7-2\frac{5}{6}=4\frac{1}{6}$$

$$\frac{3}{8}+4\frac{1}{6}=\frac{9}{24}+4\frac{4}{24}=4\frac{13}{24}\ \textit{Ans.}$$

The argument for this type of subtraction is that the "whole business of borrowing" is obviated. Whether this method is superior to the older plans is a question to be answered in the future after scientific experimentation.

Cautions and suggestions in teaching addition and subtraction of fractions. It is important that examples be arranged horizontally as well as vertically, thus: $\frac{2}{3}+\frac{5}{6}=?$, $\frac{5}{6}-\frac{2}{3}=?$, $\frac{2}{3}+?=1\frac{1}{6}$, $?+\frac{5}{6}=1\frac{2}{3}$. Children frequently confuse $\frac{3}{4}+\frac{2}{3}$, with $\frac{3}{4}\times\frac{2}{3}$. Hence practice in both vertical and horizontal algorism is a decided help.

Teachers are advised not to teach any of the forms of reduction of fractions until the need for reducing arises, either in a routine activity or in addition or subtraction of fractions. The reason for this injunction is quite apparent: motivation is assured as the child perceives the need of the process of reduction. Let us assume that the child can add $\frac{3}{4}$ and $\frac{5}{8}$ because the common denominator is in one of the addends. The new example requires the addition of $\frac{3}{4}$ and $\frac{5}{6}$. Clearly the common denominator is not visible and must be sought. Reduction to the least common denominator by inspection must now be taught. The original example in addition is set aside while the new technique is developed. But what really happens? The example that initiated and motivated the process of reduction to the least common denominator is forgotten, and the child is practicing finding the L.C.D.'s by inspection until he becomes reasonably expert at it. Experience reveals the fact that children do not find this practice unpleasant if they understand the nature of the process and the reasons for learning it.

Addition and subtraction should not be taught successively. There is every reason for having children apply to subtraction the skill they learn in adding fractions. If the child can add $\frac{4}{5}$ and $\frac{2}{3}$, he can subtract $\frac{2}{3}$ from $\frac{4}{5}$; if he can add $12\frac{5}{8}$ and $4\frac{1}{6}$, he can subtract $4\frac{1}{6}$ from $12\frac{5}{8}$. Paralleling these two processes gives greater scope to technical procedures like reducing to common denominator and to lowest terms.

Daily four-minute drills in addition and subtraction and in processes that enter into these two operations should be given for many weeks. On a chart hung before the class examples like the following should be presented:

1. $\frac{1}{4}+\frac{1}{2}=?$ $\frac{5}{8}+\frac{3}{4}=?$ $1-\frac{1}{8}=?$ $1-\frac{4}{4}=?$ $0+\frac{7}{8}=?$
   $\frac{1}{4}+\frac{1}{8}=?$ $\frac{5}{8}+\frac{3}{8}=?$ $5-\frac{5}{8}=?$ $\frac{5}{8}+0=?$ $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=?$
   $\frac{3}{4}+\frac{1}{8}=?$ $\frac{5}{8}-\frac{1}{8}=?$ $3-\frac{3}{4}=?$ $\frac{5}{8}-0=?$ $\frac{5}{2}-2=?$

2. Which fractions are equal to $\frac{1}{2}$? to $\frac{1}{8}$? to $\frac{1}{4}$?
   $\frac{2}{4}$, $\frac{4}{10}$, $\frac{5}{15}$, $\frac{2}{8}$, $\frac{3}{6}$, $\frac{3}{12}$, $\frac{4}{12}$, $\frac{6}{12}$, $\frac{3}{6}$, $\frac{2}{6}$, etc.

3. Which fractions are in lowest terms?
   $\frac{3}{6}$, $\frac{3}{5}$, $\frac{3}{15}$, $\frac{7}{8}$, $\frac{5}{10}$, $\frac{4}{12}$, $\frac{5}{12}$, $\frac{5}{6}$, $\frac{8}{12}$, $\frac{7}{12}$, etc.

4. When the denominators are	The l.c.d. is
a. $\frac{\quad}{3}$ and $\frac{\quad}{4}$	a. $\frac{\quad}{12}$
b. $\frac{\quad}{2}$ and $\frac{\quad}{3}$	b. ____
c. $\frac{\quad}{4}$, $\frac{\quad}{8}$	c. ____
d. $\frac{\quad}{4}$, $\frac{\quad}{3}$, $\frac{\quad}{8}$, etc.	d. ____ etc.

5. Give the simplest form for each of the following, thus $\frac{3 \times 2}{4 \times 2} = \frac{6}{8}$

   $\frac{3 \times 3}{5 \times 3} = ?\ \frac{8 \div 4}{12 \div 4} = ?\ \frac{2 \times 4}{3 \times 4} = ?\ \frac{5 \times 5}{9 \times 5} = ?\ \frac{12 \div 3}{15 \div 3} = ?$

6. Continue each of these lines of fractions:

   $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{6}{12} = \frac{8}{6} = \frac{5}{10} =$

   $\frac{1}{3} = \frac{3}{9} = \frac{2}{6} =$

   $\frac{1}{4} = \frac{2}{8} =$

   $\frac{3}{4} = \frac{9}{12} =$

7. Supply the missing term in each of these incomplete fractions:

   $1/3 = 2/6 = 3/? = ?/12 = 5/? = ?/18$

   $1/4 = 2/8 = 3/? = ?/16 = ?/12 = 4/?$ etc.

8. Give the answers to the following:

   $\frac{3}{4} - \frac{1}{2}$; $\frac{3}{4} - \frac{1}{8}$; $\frac{3}{4} - \frac{3}{8}$, etc.

9. $5\frac{5}{8} - 1\frac{1}{8} = ?$ $5\frac{5}{8} - 1\frac{1}{4} = ?$ $5\frac{5}{8} - 1\frac{3}{4} = ?$

10. $7 - 3\frac{1}{4} = ?$ $12 - 5\frac{5}{8} = ?$ $15\frac{3}{8} - 7 = ?$ $21 - 1\frac{1}{8} = ?$ $21\frac{1}{8} - 1 = ?$

11. $9\frac{1}{4} + ? = 12\frac{5}{8}$; $? + 5\frac{1}{3} = 15\frac{5}{6}$; $15\frac{5}{6}$ is how much greater than $12\frac{1}{3}$. Take $12\frac{5}{6}$ from $15\frac{5}{6}$.

The practice material set forth above is planned not for one lesson but for five. Daily drill periods, not to exceed five minutes, insure the functioning of this knowledge as habit, and keep habits alive.

Errors in addition and subtraction of fractions and corrective measures. We can be most helpful to pupils when we know the specific errors they make. Only then can we give them the differentiated practice they need to overcome personal weaknesses. We turn therefore to the common errors that children make.

IN ADDITION

The Error	*The Corrective Measures*
1. Errors due to multiplying both numerators and denominators, $\frac{1}{3} + \frac{1}{4} = \frac{1}{12}$, $\frac{2}{3} + \frac{3}{4} = \frac{6}{12}$. Morton[1]	1. (*a*) Use more than 2 addends as soon as possible; (*b*) emphasize the distinctiveness of each of the two processes, addition and mul-

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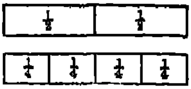
[1] R. L. Morton, *Teaching Arithmetic in the Intermediate Grades* (Silver Burdett & Co., 1927), p. 138.

The Error	*The Corrective Measures*
found that almost 45% of all errors in addition of fractions are of this type.	tiplication; (*c*) encourage oral solution with charts in full view showing functional parts and their relations.
2. Errors due to adding both numerators and denominators, $\frac{1}{3}+\frac{1}{4}=\frac{2}{7}$. About 30% of the errors in addition of fractions are of this type.	2. Suggestion (*c*) under 1, will help the child realize the error of his procedure:  Thus, $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$ becomes absurd when $\frac{1}{2}$ and $\frac{1}{4}$ are seen in their true relationship in the diagram.
3. Errors due to slips in computation. *a.* $\frac{2}{3}+\frac{3}{4}=$ $\frac{8+9}{12}=\frac{15}{12}=1\frac{3}{12}=1\frac{1}{4}$ *b.* Incorrect L.C.D. $\frac{2}{3}+\frac{3}{4}+\frac{5}{6}$, L.C.D., 18 4 into 18 = 4 *c.* Incorrect reduction to L.C.D. $\frac{2}{3}=\frac{8}{12}$; $\frac{3}{4}=\frac{9}{12}$; $\frac{5}{6}=\frac{10}{12}$	3. *a.* Point out the seriousness of these slips in elementary number work, 8 + 9 = 15, 4 into 18 = 4, etc. Make the children realize how much is lost by such petty errors. *b.* Urge greater attention to checking work and reworking each step in an example. *c.* Insist on accuracy but within reasonable bounds. No adult is free from such small errors, hence we must expect a margin of error and never harass children about these slips. Demand the exercise of reasonable care but not more.
4. Errors due to gross carelessness, misreading of signs, or faulty algorism. $\frac{2}{3}=\frac{8}{12}$ $\underline{\frac{3}{4}=\frac{9}{12}}$ $\frac{29}{12}$ (8 + 12 + 9; a denominator is erroneously included)	4. Suggestions *a*, *b*, and *c* under 3 apply with equal force to the present situation. Conditions 3 and 4 are not mutually exclusive. Practice in careful writing of numbers and symbols so that a plus sign does not look like a multiplication sign, will help reduce errors due to the misreading of semi-legible forms.

These four classes of errors do not sum up all those that children make. Occasionally denominators are omitted, and the sum of the numerators, 7, becomes the startling answer to $\frac{3}{5}+\frac{4}{9}$. Sometimes numerators are added and denominators are multiplied, giving $\frac{3+4}{5\times 9}$ or $\frac{7}{45}$ as the sum of $\frac{3}{5}$ and $\frac{4}{9}$. Other unexpected mistakes are found but they are not common to a significantly large group of pupils.

IN SUBTRACTION

Various analyses have been made in which the process of subtracting fractions was reduced to a large number of unit skills—some investigations have yielded as many as forty-three. Among these unit skills are reducing to a common denominator, converting a unit taken from a whole number into a fraction when "borrowing" is necessary, adding to the fraction in the minuend the fraction $\frac{8}{8}$ or $\frac{6}{6}$ obtained by converting a unit, etc.[2] To teach each of these unit skills would require more time than the schools can give to insure mastery of arithmetic. Analysis of a complex arithmetical process into its constituent unit skills is extremely helpful, but the analysis must not be too fine. Perfect mastery of any process in any one grade cannot be attained. Theoretically, children may err in any of the forty-three unit skills and in actual practice they do; but the teacher's concern must be centred on those skills which large groups of children cannot master easily. We shall list only the common errors and ask the reader to formulate the corrective measures, a task which is simple enough after the analysis of the corrective measures for errors in addition.

1. Errors due to subtracting the numerators and the denominators or only the numerators or only the denominators. To illustrate: $\frac{3}{5}-\frac{1}{3}=\frac{2}{2}=1$; $3\frac{5}{8}-2\frac{3}{4}=1\frac{2}{4}=1\frac{1}{2}$; $\frac{3}{4}-\frac{1}{3}=\frac{2}{12}$;

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[2] Knight, Luse, and Ruch, *Problems in Teaching Arithmetic* (Iowa City, Iowa Supply Store, 1924).

$\frac{5}{8} - \frac{1}{4} = \frac{5}{4}$. About half the errors in subtraction of fractions are of this type.[3]

2. Errors in computation. The children may know the entire process of subtraction of fractions but they nevertheless may be guilty of $9 - 5 = 3$. Thus $\frac{5}{6} - \frac{1}{4}$ may be turned into $\frac{10}{12} - \frac{3}{12} = \frac{13}{12}$ or $\frac{8}{12}$ or any number of twelfths. About 10 per cent of the errors are due to this weakness.
3. Errors due to confusion of procedures. In many instances pupils divide instead of subtract. Examples: $\frac{3}{5} - \frac{1}{3}$ may produce $\frac{9}{5}$ when minus is read as divide.
4. Errors in finding the least common denominator or in reducing a fraction to the least common denominator, and failure to reduce.
5. Failure to reduce a first answer to lowest terms.
6. Errors due to "borrowing."
    - *a.* $5\frac{1}{8} - 2\frac{3}{4} = 2\frac{1}{4}$. Here the child forgot the $\frac{1}{8}$ in the minuend.
    - *b.* $5\frac{1}{8} - 2\frac{3}{4} = 2\frac{6}{8}$. Here the pupil thought $4\frac{9}{8} - 2\frac{3}{4} = 2\frac{6}{8}$; he forgot to think of $\frac{3}{4}$ as $\frac{6}{8}$.
    - c. $5\frac{1}{8} - 2\frac{3}{4} = 3\frac{3}{8}$. Here the child forgot that he converted one unit into $\frac{8}{8}$.
    - *d.* $5\frac{3}{4} - 2\frac{1}{8} = 2\frac{5}{8}$. The borrowing idea persisted even when unnecessary.

In planning a diagnostic test in subtraction of fractions examples must be included that are designed to reveal the child's ability to avoid the common errors. Persistent failure in one of the types of subtraction listed above is adequate reason for prescribing differentiated and individualized drill.

MULTIPLICATION AND DIVISION OF FRACTIONS

Gradation. In multiplication we have the simplest of the four operations with common fractions. To multiply $\frac{2}{3}$ by $\frac{3}{4}$ is decidedly less involved than adding these two fractions or subtracting the smaller from the larger. Here we find the justification of the growing practice of teaching multiplication of common fractions before the other fundamental operations with fractions.

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[3] R. L. Morton, *Teaching Arithmetic in the Intermediate Grades* (Silver Burdett & Co., 1927), p. 161.

Experience teaches that the most useful procedure in multiplication of common fractions is to multiply the numerators for the numerator in the answer and to multiply the denominators for the denominator in the answer. This rule cannot be taught by the examples which most textbooks and teachers employ to introduce multiplication of common fractions, viz., 4×½ and ½ of 4. Surely these types of examples do not call for the application of the rule for multiplying common fractions. Children who understand the primary concept of fractions know that 4×½, like 4× any quantity, produces four times one half or 4/2. They know from their early number work that ½ of 10 cents is 5 cents and ½ of 4 is 2—results obtained by division, not by multiplication. The forms, 4×½ and ½ of 4, are both very important and should be taught thoroughly but we must recognize that they will not lead to the rule for multiplying a fraction by a fraction. Not until we present ½×⅕ or ½ of ⅕ do the children meet a situation that calls for the fundamental rule in the multiplication of common fractions.

No extended gradation is necessary in multiplication. Children have difficulty in getting the intent of ½×½ but understand ½ of ½ more readily. The *of* should precede the × in multiplication of common fractions. We suggest therefore, the following steps in gradation:

Progressive Steps	*Justification of the Gradation*
1. ½ of 4; ⅓ of 9. 4 × ½ or ½ × 4. 5 × ⅓ or ⅓ × 5.	1. Old knowledge is reviewed here to make sure that these types are understood and solved readily. If children experience any difficulty, revert to concrete aids. Thus, arrange 4 dots, ... into 2 groups of the same size; . . . . how many in each? 2 is what part of all the dots? ½ of 4 is what?

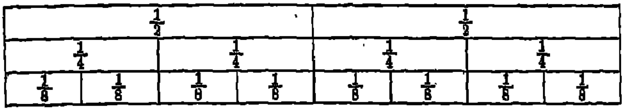
Progressive Steps.	*Justification of the Gradation*
2. ⅓ of 9 = ? Then ⅔ of 9 = ? ¼ of 8 = ? Then ¾ of 8 = ?	2. Technically we have not yet introduced the multiplication of common fractions. We are simply carrying step 1 to a higher level of skill.
3. ½ of ½; ½ of ¾; ⅓ of ½	3. The *of* precedes the *times* idea. A fraction of a fraction presents two numerators and two denominators, so that the practice of multiplying numerators and denominators can be seen more clearly.
4. ½ × ½; ⅓ × ¾	4. The ideas in steps 2 and 3 are applied to the multiplication form.
5. 2½ × ¼; ½ of 2½	5. Mixed numbers are introduced.
6. ⅔ × 9; ¾ of 24	6. An integer multiplied by a fraction. The implied but unexpressed denominator of the integer is the new difficulty.
7. ⅔ × ⅘ × ¾ = ? ¾ of ⅚ of ⅔ = ?	7. More than two factors are introduced.
8. 1⅔ × ⅘ × 1½ = ?	8. Step 7 applied to mixed numbers.

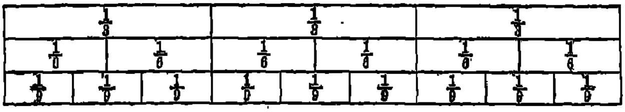
Teaching multiplication of fractions. The key-note in these lessons is concreteness. Paper folding, charts with suitable subdivisions, groups of objects that lend themselves to rearrangement (for example, ½ of 8 pupils) are all very helpful.

To teach graded step number 3, a fraction of a fraction, use diagrams well drawn with variously colored lines indicating the different fractional parts. The children are asked to find the answers to the following by consulting the charts:

½ of ½
½ of ¼
½ of ¾

½ of ⅔
¼ of ⅘
½ of ⅓





In each case the children go to the diagram to find the answer. The teacher then suggests the need of finding the results without recourse to any objective aid. With this aim, each example is taken up and subjected to the following analysis:

½ of ½=¼; what must be done with the two 1's of the numerators to obtain 1 in the numerator of the answer? If we multiply the two 2's of the denominators what do we obtain? ½ of ¾=⅜; what must be done with the 3 and the 1 to obtain 3 for the numerator in the result? What must be done with the 2 and the 4 to obtain 8, the denominator of the result? After similar analyses of ½ of ½, ½ of ¼, ½ of ⅜, the class is led to formulate the law and to test it by applying it to ½ of ⅔, ¼ of 4/9, ⅕ of 5/9.

A few days later the multiplication form is presented with the aid of the diagrams. I have $½, my friend has ½ times as much; what has he? If I have $½, and my friend has ¼ times as much, what has he? etc.

$½ × ½ = $¼	$⅔ × ½ = $⅓
$¼ × ½ = $⅛	$4/9 × ¼ = $⅑
$¾ × ½ = $⅜	$5/9 × ⅕ = $⅑

The class is now shown the similarity in computation between ½ of ½ and ½×½ and is led to recognize that *of* means *times* in multiplication of fractions. The law is tested by applying it to examples of the type, 4/9×¼, which yields 4/36; this when reduced becomes ⅑. Sufficient practice must be given to insure the function of the rule as habit.

The new ratio in multiplying by a fraction. The chief difficulty in multiplication and division by fractions, it is held, lies not in cancellation nor in inverting the divisor, but in the new relationships that are set up. For years the child has seen × produce a larger and ÷ a smaller quantity. Now the ratio is reversed. Teachers are, therefore, advised by some writers to display the following statements[4] to accompany the exercises suggested below:

When you multiply a number by anything more than 1 the result is larger than that number.
When you multiply a number by 1, the result is the same as that number.
When you multiply a number by less than 1, the result is smaller than that number.

Exercises

$16 \times 2 =$	$12 \times 2 =$
$8 \times 2 =$	$6 \times 2 =$
$4 \times 2 =$	$3 \times 2 =$
$2 \times 2 =$	$1 \times 2 =$
$1 \times 2 =$	$\frac{1}{3} \times 2 =$
$\frac{1}{2} \times 2 =$	$\frac{1}{6} \times 2 =$
$\frac{1}{4} \times 2 =$	$\frac{1}{12} \times 2 =$
$\frac{1}{8} \times 2 =$	

When you divide a number by anything more than 1, the result is smaller than that number.
When you divide a number by 1, the result is the same as that number.
When you divide a number by anything less than 1, the result is larger than that number.

$16 = \underline{\quad 2 \quad}$ 8's	$16 \div 8 = 2$	$24 = \underline{\quad 2 \quad}$ 12's	$24 \div 12 = 2$
$16 = \underline{\qquad}$ 4's	$16 \div 4 = 4$	$24 = \underline{\qquad}$ 6's	$24 \div 6 = 4$
$16 = \underline{\qquad}$ 2's	$16 \div 2 = 8$	$24 = \underline{\qquad}$ 3's	$24 \div 3 = 8$
$16 = \underline{\qquad}$ 1's	$16 \div 1 = 16$	$24 = \underline{\qquad}$ 2's	$24 \div 2 = 12$
$16 = \underline{\qquad}$ $\frac{1}{2}$'s	$16 \div \frac{1}{2} = 32$	$24 = \underline{\qquad}$ 1's	$24 \div 1 = 24$
$16 = \underline{\qquad}$ $\frac{1}{4}$'s	$16 \div \frac{1}{4} = 64$	$24 = \underline{\qquad}$ $\frac{1}{2}$'s	$24 \div \frac{1}{2} = 48$
		$24 = \underline{\qquad}$ $\frac{1}{3}$'ds	$24 \div \frac{1}{3} = 72$
		$24 = \underline{\qquad}$ $\frac{1}{4}$'s	$24 \div \frac{1}{4} = 96$

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[4] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), pp. 79–80.

Experienced teachers frequently challenge the prominence given to these inverse ratios in multiplication and division by fractions. They point out that only the brightest of their children see the changed relationships and are concerned enough to ask about them; that any analysis of the new order tends to confuse the less capable students; that, like many axioms of elementary mathematics, these inverse ratios become quite obvious with maturity. We have before us opposing estimates of a teaching situation based on the logic of the subject-matter and on actual experience. In the absence of conclusive data the reader must form his own evaluation of the difficulty which the reversed ratios produce.

Common errors in multiplication of fractions. We must now turn to a consideration of five sources of difficulty which children experience in the multiplication of common fractions. The remedial measures are obvious for they are clearly implied in the causes of these common errors.

1. Errors due to wrong operation. Morton[5] reports that 33.8 per cent of the errors revealed in his study of children's work were due to the performance of the wrong operation. Thus ⅔ × ¾ gave 17/12 or 1 5/12 because the two fractions were added.
2. Errors due to failure to reduce answers to lowest terms. In ⅔ × ¾ the answer submitted is 6/12, not ½.
3. Errors due to faulty computation. Here we find the slips that go back to basic computations of whole numbers. Thus, ⅖ × 4/7 becomes 8/30 because the child thought that 30 is the product of 5 and 7. These three classes of errors account for about 80 per cent of the mistakes made by the children examined by Morton.
4. Errors due to confusion of 1's in cancellation. $\overset{1}{\cancel{2}}/\underset{1}{\cancel{3}} \times \overset{1}{\cancel{3}}/\underset{4}{\cancel{8}}$ becomes 0/4 and finally 4, because the pupil regarded his 1's as numbers to be discarded.
5. Errors due to failure to use the fraction. Thus in 24 × 3⅜ or 24⅜ × 3, the child obtains 72⅜. It is clear that the fraction in either the multiplicand or the multiplier is disregarded.

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[5] R. L. Morton, *Teaching Arithmetic in the Intermediate Grades* (Silver Burdett & Co., 1927), p. 175.

The five common errors are not difficult to eliminate. The corrective measures have already been presented with suitable illustrations in the discussion of common errors in other fundamental operations.

CANCELLATIONS

Gradation. In the teaching of cancellation, gradation is very important. This can be seen from the following analysis:

Steps in Gradation	*Justification for the Gradation*
1. $\frac{49}{56}=\frac{7\times 7\div 7}{8\times 7\div 7}=\frac{7}{8}$	1. Here we have a review of the steps that lead to the reduction of lowest terms.
2. $\frac{5}{8}\times\frac{3}{5}$, $\frac{5}{8}$ of $\frac{3}{5}$, $\frac{5\times 3}{8\times 5}$	2. The numbers to be cancelled are both seen (5 and 5).
3. $\frac{2}{5}\times\frac{1}{6}$, $\frac{2}{5}$ of $\frac{1}{6}$, $\frac{2\times 1}{5\times 6}$	3. The common divisor, 2, is seen, but the results are not 1 and 1 as above.
4. $\frac{2}{3}$ of $\frac{3}{6}$, $\frac{2}{3}\times\frac{3}{6}$, $\frac{2\times 3}{3\times 6}$	4. Combination of steps 2 and 3.
5. $\frac{3}{10}\times\frac{4}{6}$ of $\frac{5}{8}$	5. Conditions of step 4 applied to many fractions.
6. $\frac{4}{7}\times\frac{5}{6}$, $\frac{4}{7}$ of $\frac{5}{6}$	6. The common divisor, 2, is not a numerator or a denominator.
7. $\frac{4}{5}\times\frac{5}{6}$ of $\frac{9}{10}$	7. Application of previous steps to more than two fractions.
8. $1\frac{2}{3}\times\frac{1}{5}\times 15\times\frac{7}{10}$	8. Introduction of mixed numbers.

Mode of rationalization. Review the law of reduction of fractions to lowest terms then present $\frac{5}{8}\times\frac{3}{5}$. Not knowing how to cancel, the class solves as follows:

$$\frac{5}{8}\times\frac{3}{5}=\frac{15}{40}=\frac{3}{8}$$

The teacher then shows that the children performed the following operations:

$$\frac{5}{8}\times\frac{3}{5}=\frac{5\times 3}{8\times 5}=\frac{15\div 5}{40\div 5}=\frac{3}{8}$$

The class may now be required to solve,

$$\begin{array}{l|l} 2\times 4\div 2=? & 16\times 8\div 4=?\\ 2\div 2\times 4=? & 16\div 4\times 8=? \end{array}$$

Finding that the results are the same, we elicit (1) that the processes of division and multiplication may be interchanged; and (2) that it is desirable to perform the division first because smaller numbers are thus obtained, $16\times8\div4=124\div4=32$, but $16\div4\times8=4\times8=32$.

The children are now led to realize that in their solution of $5/8\times 3/5$ they followed the longer method; hence the division by 5 should precede the multiplication. They are now shown the form,

$$\frac{\overset{1}{\cancel{5}}}{8}\times\frac{3}{\underset{1}{\cancel{5}}}=\frac{1\times 3}{8\times 1}=\frac{3}{8}\quad\text{or better}\quad\frac{\overset{1}{\cancel{5}}}{8}\times\frac{3}{\underset{1}{\cancel{5}}}=\frac{3}{8}$$

This rationalization should not be forced. Children who have enough mathematical insight to perceive that cancellation is an attempt to reduce fractions to lowest terms before the final multiplication should be led to a full understanding of the process. It may be found necessary to teach cancellation to other children on the basis of sheer authority. Since the aim is to develop habits of rapid and accurate factoring, no undue effort should be made to rationalize this process for all children.

General suggestions for teaching cancellation. Every effort must be made to impress the factoring idea or the dividing process on the minds of the pupils. Children should be required to indicate the quotient after each act of factoring. Thus, $\frac{3}{\cancel{5}}\times\frac{\cancel{5}}{8}$ should not be permitted; the longer form $\frac{3}{\underset{1}{\cancel{5}}}\times\frac{\overset{1}{\cancel{5}}}{8}$ is more desirable. When quotients are not written children develop the idea that cancellation is " crossing out " and in the example 3/5 × 5/3 they proceed to eliminate all numbers, thus $\frac{\cancel{3}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{3}}$, and give us the answer, zero. But when the work shows quotients as in $\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{5}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{3}}}$, the child is more likely to see that the answer is $\frac{1}{1}$ or 1.

Cancellation must be used in every variety of arithmetical situation. In solving problems of unitary analysis, or by ratio methods, successive multiplications and divisions should always be organized into fractional forms so that cancellations may be made. In problems involving areas, denominate numbers, and interest, the cancellation method is usually superior to successive multiplication and divisions. This is seen in the following solutions:

1. Find the interest on \$2500 for 72 days at 6%. 72/360 of 6/100 × \$2500 = interest.
2. How many jars holding 3 pints each can be filled from a barrel containing 24 gallons? 24 × 4 × 2 = pints in bbl, hence $\frac{24 \times 4 \times 2}{3}$ = no. of jars that can be filled.

Division of Fractions

Teachers ascribe many reasons for their belief that division by a fraction is the most difficult of the fundamental operations. They argue that the process of inversion is one cause of this difficulty; that another cause is the increasing quotient when the divisor is a fraction whose value is less than one. The first contention is not borne out by the results of any inquiry. Nor have we data to substantiate the second reason. Slow children who experience difficulty in arithmetic do not see that quotients increase, not decrease, with certain fractional divisors. Bright children get this inverse relationship themselves.

The real difficulty lies in the fact that children either fail to invert at all, or invert the dividend, not the divisor. The cause is clear enough; they do not understand the division idea in whole numbers and therefore do not understand it in fractions. Pupils in the junior high school frequently divide by the smaller of two whole numbers. In the example, "If 4 acres are to be divided into 12 building plots, what will the area of each plot be?", the answer, 3 is given with surprising frequency. We must teach the division idea through a variety of concrete experiences and rapid number drills:

1. Paper folding to illustrate examples like the above.
2. Subdivision of lines, rectangles and circles to illustrate the division idea.
3. Daily two minute drills like the following:

a. There are ___ 3's in 24
___ 4's in 24
___ 6's in 24
___ 8's in 24
___ 12's in 24

There are ___ 3's in 26
___ 4's in 26
___ 5's in 26
___ 6's in 26
___ 7's in 26
___ 9's in 26

b. 12 ÷ 4 = ?   12 ÷ 3 = ?   12 ÷ ? = 4   12 ÷ ? = 3
? ÷ 4 = 3   ? ÷ 3 = 4
42 ÷ 7 = ?   42 ÷ 6 = ?   42 ÷ ? = 7   42 ÷ ? = 6
? ÷ 7 = 42   ? ÷ 6 = 42

c. Divide 72 by 9; the result is ___.
72 divided by 9; the result is ___
Divide 81 by 9; the result is ___.
81 divided by 9; the result is ___

d. 8 goes into 72 ___ times
9 goes into 54 ___ times

e. Which is the divisor?
7 into 63 = ?   63 ÷ 7 = ?   Divide 49 by 7.
At 4 cents each, how many can I buy for 36 cents?
96 cents will buy how many 12 cent painting brushes?
A car runs about 16 miles on one gallon of gasoline; about how many gallons will be required for a trip of 256 miles?

f. Which is the divisor?
½ ÷ 2; 2 ÷ ½; ½ into 2; 2 into ½; divide 2 by ½; divide ½ by 2.

g. Practical problems, in fractions, involving division. Let children identify the divisor.

Once a child has grasped the division idea he will not multiply to find the answer nor will he be in doubt which one of the factors or numbers to invert. Two-minute practice daily in identifying the divisor in variously worded examples will go far to develop accuracy in division by a fraction. The assumption, of course, is that these pupils have mastered multiplication of fractions and cancellation.

Gradation. Nice gradation in division by fractions is not important for teaching. The obvious progression will answer all practical purposes. Naturally, one begins to teach division of fractions with proper fractions, and introduces integers and mixed numbers in later lessons. Many authorities prefer to begin with the division of a fraction by an integer. The question is open to dispute. Experience seems to show that the most effective method of division by fractions is to invert the divisor and multiply. The sooner the child learns this rule and to identify the divisor, the sooner he acquires the necessary skill in the division of fractions.

Rationalization. Many plans of teaching division of fractions are advocated, but none has been established through scientific experimentation. We must examine these teaching procedures with a full realization that at present personal opinion is the only criterion of selection.

Through reduction to similar denominators. Let us assume that $2/3$ is to be divided by $2/9$. The children are led to formulate the successive steps in the equations, $2/3 \div 2/9 = 6/9 \div 2/9 = 6 \div 2 = 3$. This mode of teaching division by a common fraction is generally clear, simple, and comprehensible to children, but it does not teach the very desirable law of inversion. In simple fractions like $2/3$ and $2/9$, $3/4$ and $1/8$, this process of reducing to similar denominators is not inconvenient, but when fractions with larger denominators and mixed numbers are introduced, the reduction to similar denominators becomes involved and uneconomical.

Through analysis of the division form. A second method is to analyze the steps in dividing by any number, whole or fractional, and then to formulate the process common in all division. The steps in this method are summarized below:

a. $\begin{array}{l}\;\;6\\2\overline{)12}\end{array}$, that is, same as $\frac{1}{2}$ of 12 or $12 \times \frac{1}{2}$

b. $\begin{array}{l}\;\;12\\1\overline{)12}\end{array}$, that is, divisor decreases, quotient increases.

c. $\begin{array}{l}\;\;\;24\\\frac{1}{2}\overline{)12}\end{array}$, that is, 2 halves in 1, in 12 there are 12×2 or 24 halves, hence $\frac{1}{2}\overline{)12}$ or $12 \div \frac{1}{2}$ is the same as 12×2.

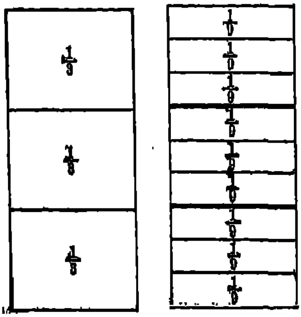
d. $\frac{3}{2}\overline{)12}$, to reduce 12 to halves, multiply 12 by 2, $\frac{3}{2}\overline{)\;\;(12 \times 2)}$ halves or 3 halves$\overline{)\;\;(12 \times 2)}$ halves, equal $3\overline{)\;\;(12 \times 2)}$ or $3\overline{)24}$; hence $\frac{3}{2}\overline{)12}$ or $12 \div \frac{3}{2} = 12 \times 2 \div 3$ or $12 \times \frac{2}{3}$; clearly $12 \div \frac{3}{2} = 12 \times \frac{2}{3}$.

A comparison is now made of steps *a*, *b*, *c*, and *d*, and the conclusion, invert the divisor, is evolved. Although the process is logical, step by step, it is a method that is too verbal to insure comprehension by immature pupils. These progressive steps usually do nothing more than prompt children to accept the preliminary conclusions that lead to the law of inversion. The whole lesson resolves itself into an imposition of adult reasoning on children too young to react intelligently.

By reducing the divisor to one. A simple rationalization, but one open to the same objections, is found in the method which reduces the divisor to one. The children are shown that to divide by 1 leaves the quantity unchanged, hence $\begin{array}{l}\;\;8\\1\overline{)8}\end{array}$, $7 \div 1 = 7$, and $\frac{2}{3} \div 1 = \frac{2}{3}$. To divide $\frac{2}{3}$ by $\frac{3}{4}$, the form $\frac{3}{4}\overline{)\frac{2}{3}}$ is used. The children are now asked to tell by what to multiply $\frac{3}{4}$ to produce 1. If they do not give the correct answer the teacher supplies the information, namely, $\frac{4}{3}$. Since a division example is only a fraction, the divisor and the dividend of $\frac{3}{4}\overline{)\frac{2}{3}}$ may be multiplied by the same number, hence $\frac{3}{4}\overline{)\frac{2}{3}}$ becomes $\frac{3}{4} \times \frac{4}{3}\overline{)\frac{2}{3} \times \frac{4}{3}}$, and then $1\overline{)\frac{2}{3} \times \frac{4}{3}}$, and finally, $1\overline{)8/9}$ or $8/9$. It is evident from the above that $2/3 \div 3/4 = 2/3 \times 4/3$. A few more examples will give a basis for generalizing. But the experienced teacher sees in this method a means of rationalizing for the exceptionally gifted child rather than for the average pupil.

By objective demonstration and parallel correspondence. A practical situation was presented as the introductory step to motivate division by a fraction. Class badges are to be made from a ribbon $10\frac{1}{2}$ yards long; if $3/4$ yd. is allowed for each, will there be enough badges for all? When some children suggested that the ribbon should be cut into small lengths of $3/4$ yd. each, others objected on the ground that if the number were found insufficient, it would be too late to make the badges smaller. It was then decided "to figure it out first," and the situation was reduced to $10\frac{1}{2} \div 3/4$ or $21/2 \div 3/4$. The children then realized that this was, for them, a new process in arithmetic. To the teacher's question, "What should we learn in arithmetic to-day?" the ready answer was given: "To divide by a fraction."

A double diagram was presented, each rectangle of similar dimensions, and the corresponding parts were emphasized by colored chalk. From these diagrams the children were asked to find the answers to the following:



$1/3 \div 1/9$, $2/3 \div 1/9$, $2/3 \div 2/9$.

The children found by aid of the diagram

$$1/3 \div 1/9 = 3$$
$$2/3 \div 1/9 = 6$$
$$2/3 \div 2/9 = 3$$

The following facts were elicited:

$4 \div 2 = 2$; $1/2$ of 4 or $4 \times 1/2 = 2$.
$8 \div 4 = 2$; $1/4$ of 8 or $8 \times 1/4 = 2$.
$8 \div 2 = 4$; $1/2$ of 8 or $8 \times 1/2 = 4$.

The children were then led to formulate the law that instead of dividing by a whole number we may multiply by the divisor inverted. The class was then asked, "If this holds true for whole numbers, will it hold for fractions?" The children said they did not know, but they would try the process with the fractions above, because they knew the correct answers. The next step appeared on the blackboard as follows:

By diagram: $\frac{1}{3} \div \frac{1}{9} = 3$ $\quad \frac{2}{3} \div \frac{1}{9} = 6$ $\quad \frac{2}{3} \div \frac{2}{9} = 3$
By rule: $\frac{1}{3} \times \frac{9}{1} = 3$ $\quad \frac{2}{3} \times \frac{9}{1} = 6$ $\quad \frac{2}{3} \times \frac{9}{2} = 3$

Every child concluded that the rule held true in both fractions and whole numbers and that there was "nothing new in division of fractions, for you turn the divisor upside down and then multiply." The original problem was now recalled, $10\frac{1}{2}$ yd $\div \frac{3}{4}$, or $\frac{21}{2} \div \frac{3}{4}$, and it was solved by inverting the $\frac{3}{4}$ and then multiplying,

$$\frac{\overset{7}{\cancel{21}}}{\underset{1}{\cancel{2}}} \times \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{3}}} = \frac{14}{1} = 14, \text{the number of badges.}$$

The applications must be varied in form and plentiful in number until the child can divide by common fractions in any reasonable example. The drills should include such examples as

$1 \div \frac{1}{2}$,
$1 \div \frac{1}{4}$,
$1 \div \frac{1}{5}$,
$\frac{1}{2} \div \frac{1}{2}$,
$\frac{1}{2} \div \frac{1}{4}$,
$\frac{1}{4} \div \frac{1}{2}$,
$\frac{1}{3}\overline{)\frac{3}{4}}$,
$5\overline{)\frac{5}{8}}$,
$3\overline{)\frac{3}{8}}$,
$\frac{1}{3}\overline{)\frac{3}{8}}$,
$3 \div \frac{1}{3}$,
$\frac{1}{3} \div 3$,
$3 \div \frac{1}{2}$,
$2 \div \frac{1}{2}$,
$\frac{5}{12} \div 1\frac{2}{3}$, etc.

These examples may be put on perception cards and the drills given, at first daily, and then in spaced intervals until children develop a sense of sureness. There can be no excuse for finding pupils in the last two years of the school course in hopeless confusion when they are asked to find the answers to the following without recourse to paper computation:

$2 \div \frac{1}{4}$, $\frac{1}{4} \div 2$, $\frac{1}{3} \div \frac{1}{6}$, $\frac{1}{6} \div \frac{1}{2}$.

Common errors in division of fractions and remedial measures. There are three common pitfalls that children do not usually avoid in division of fractions. We must identify these and indicate corrective exercises for those who lack thorough mastery.

The Error	*The Corrective Measures*
1. Errors due to wrong computation, $\frac{3}{4} \div \frac{9}{2} =$ $\frac{{}^{1}\cancel{3}}{\cancel{4}_{2}} \times \frac{\cancel{2}^{1}}{\cancel{9}_{3}} = \frac{1}{5}.$	1. Children must be led to understand the full significance of these slight errors of computation. Greater care in work, and checking each step as well as the final answer will help develop added concern with simple manipulation.
2. Errors due to incorrect inversion: *a.* Failure to invert, hence the wrong process, multiplication, is used. *b.* Inversion of dividend instead of the divisor.	2. Failure to invert may be due to carelessness, to ignorance of the process of dividing by fractions, or to inability to identify the divisor. The remedy is obvious: careful teaching of the division idea as was outlined in the introduction to this discussion with exercises in identifying divisors.
3. Errors due to failure: *a.* to reduce answers to lowest terms, or *b.* to reduce improper fractions to mixed number answers.	3. By this time the children know the meaning and the process of reducing to lowest terms or to mixed numbers. They overlook these reductions because of carelessness or overconcern to be correct. Reminders written on a central section of the blackboard help to develop the habit of reducing answers.

Elaborate analyses have been made of the skills involved in division of common fractions. One of these, to which frequent reference is found, lists fifty-five discrete unit skills.

We give only part of this carefully formulated list.[6] What practical application is intended of these elaborate analyses

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[6] F. B. Knight, "Curriculum Problems and Their Scientific Solution," *Third Year Book* of the Department of Superintendence, pp. 66–69.

ANALYSIS OF DIVISION OF FRACTIONS IN TERMS OF LEARNING PROCESS

	Unit of Skill Number
I. As to the form of stating the example	
A. Fractions written in figures	
1. Indicated divisions, as $\frac{3}{8} \div \frac{1}{2}$	1
2. Words "divided by", as $\frac{4}{5}$ divided by $\frac{1}{8}$	2
3. Complex fractions, as $\frac{\frac{3}{5}}{\frac{1}{3}}$	3
4. Division indicated by parentheses, as $\frac{1}{4})\underline{\frac{1}{12}}$, or $\frac{1}{2})\overline{\frac{4}{5}}$	4
B. Fractions written with words	
1. Indicated division, as two-sevenths, one-sixth	5
2. Words "divided by" used, as one-eighth divided by three-fifths	6
II. As to Procedure	
A. Nature of terms—Expression of all terms as fractions	
1. Unit fraction ÷ unit fraction, as $\frac{1}{8} \div \frac{1}{9}$	7
2. Unit fraction ÷ other proper fraction, $\frac{1}{3} \div \frac{2}{5}$	8
3. Unit fraction ÷ improper fraction, as $\frac{1}{9} \div \frac{7}{4}$	9
B. Cancellation	
1. No cancellation possible, as $\frac{3}{5} \times \frac{4}{7}$	34
2. Single cancellation	
a. One number a factor of the other, as $\frac{3}{5} \times \frac{7}{9}$	35
b. Two numbers with a common factor, as $\frac{6}{7} \times \frac{5}{8}$	36
3. Double cancellation	
a. In each case one number a factor of the other, as $\frac{3}{4} \times \frac{8}{9}$	37
b. In each case two numbers with a common factor, as $\frac{8}{9} \times \frac{15}{18}$	36 38
C. Multiplication	
1. Neither factor unity in numerator or denominator, as $\frac{3}{8} \times \frac{5}{6}$	44
2. One factor unity in numerator, as $\frac{1}{7} \times \frac{5}{8}$	45
D. Analysis of quotient	
1. Quotient a whole number	
a. Cancellation complete—irreducible as $\frac{7}{1}$	49
b. Cancellation incomplete—reducible as $\frac{15}{3}$	50
2. Quotient a proper fraction	
3. Quotient an improper fraction—reduce to mixed number	
a. Cancellation complete—fraction irreducible, as $\frac{13}{4}$	54
b. Cancellation incomplete—fraction reducible, as $\frac{21}{6}$	55

is not revealed by their authors. Teachers and supervisors must not assume that this minute gradation is necessary nor that these fifty-five unit skills must be made the basis of diagnostic testing. A careful reading of such a list is undoubtedly stimulating to a professional-minded teacher. It helps her diagnose difficulties presented by individual pupils. It makes vivid what we teachers know: that learning is an adventure beset by many trials. Overgradation may result in encumbering rather than simplifying class teaching.

Complex and compound fractions. Compound fractions, ⅔ of 6/7 and ½ of ⅖, arise frequently in the practical applications of arithmetic. We saw their importance in our study of multiplication of fractions and of cancellation. Complex fractions that have fractional forms in numerator or in denominator or in both have no justification in the elementary-school course. When these complex fractions are prescribed by a course of study they should be taught deductively with little rationalization. The pupil should be advised to simplify the numerator and the denominator, and then translate the complex fraction into division form. To illustrate:

Simplify the following: $\dfrac{\frac{3}{4}+\frac{5}{6}}{\frac{1}{9}}$; $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{9}}$; $\dfrac{\frac{8}{9}}{\frac{2}{3}}$.

$$\dfrac{\frac{3}{4}+\frac{5}{6}}{\frac{1}{9}}=\dfrac{\frac{9}{12}+\frac{10}{12}}{\frac{1}{9}}=\dfrac{\frac{19}{12}}{\frac{1}{9}}=\frac{19}{12}\div\frac{1}{9}=\frac{19}{\underset{4}{\cancel{12}}}\times\frac{\overset{3}{\cancel{9}}}{1}=\frac{57}{4}=14\tfrac{1}{4}$$

$$\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{9}}=\dfrac{\frac{3}{6}+\frac{5}{6}}{\frac{2}{9}}=\dfrac{\frac{8}{6}}{\frac{2}{9}}=\frac{8}{6}\div\frac{2}{9}=\frac{\overset{\overset{2}{\cancel{4}}}{\cancel{8}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{6}}}\times\frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{2}}}=\frac{6}{1}=6$$

$$\dfrac{\frac{8}{9}}{\frac{2}{3}}=\frac{8}{9}\div\frac{2}{3}=\frac{\overset{4}{\cancel{8}}}{\underset{3}{\cancel{9}}}\times\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}}=\frac{4}{3}=1\tfrac{1}{3}$$

The process of simplification requires no rationalization. Each succeeding step follows the ratios inherent in these numbers. If at all possible, complex fractions should be reserved for the course in elementary algebra.

Problems for Study and Discussion

1. Inability to add fractions is usually due to one or more of four causes. In parallel vertical columns set forth, (*a*) these causes, and (*b*) corrective measures designed to overcome each of them.

2. "The child brings to class ideas concerning fractions, gained from a variety of experiences. The concept fraction may take form before the concept of whole numbers."

(*a*) Make as complete a list as possible of the experiences referred to. What inferences for classroom teaching may be drawn from this list?

(*b*) Explain, with illustrations, the meaning of the second sentence of the quotation.

3. Make a list of the technical terms and expressions found in Chapters XXV and XXVI. Which of these should the child know? Why are the others introduced? Set forth a few suggestions that should guide in teaching technical terms.

4. Invent ten problems, five in addition and five in subtraction of fractions. Submit these for class criticism. In terms of what standards do your classmates or colleagues judge problems?

5. List the skills required in the subtraction of fractions that are not acquired in the addition of fractions. Give for each such skill appropriate teaching procedures.

6. "Class medians on standardized tests on common fractions, particularly on division, are distressingly low." (*Contemporary Guide in the Teaching of Arithmetic,* p. 74.) Account for the low scores and present corrective measures.

7. *Cancel* and *cancellation* are unfortunate terms because they are not descriptive of the arithmetical process to which they refer. What is the arithmetical process? Why does the term *cancel* convey an incorrect idea? What term would be in closer keeping with the process that actually takes place? What is the bearing of all this on the teaching of cancellation?

8. "There are few practical situations that call for the division of fractions or mixed numbers by a fraction. The school should minimize and even eliminate this process." Write ten practical problems which involve division by a fraction. If necessary, consult textbooks in arithmetic. Do you agree with the quotation? Justify your answer.

9. List three technical processes in fractions which are presented in these two chapters but which are not employed in business or industry. Do you believe that they should be taught to children? Justify your answer.

10. Invent a graded series of, (*a*) exercises, and (*b*) problems designed to lead to rapid and accurate identification of the divisor, especially the fractional divisor. Why is such skill important?

11. Examine those sections in a textbook in arithmetic that are devoted to common fractions. Do you find any elements of procedure that are at variance with teaching principles presented in these two chapters? If you do, formulate your position, with adequate justification on the question at issue.

12. (*a*) List six references on common fractions. (Consult the bibliography at the end of this book.)

(*b*) Secure three or four recent volumes of these professional journals: *Journal of Educational Research*; *Journal of Educational Psychology*; *Educational Administration and Supervision*; *School Review*. Compare the space devoted to arithmetic with that given to any other school subject; state an inference, if it is justifiable. Select two or three articles that deal with some phases of arithmetic. Write a critical estimate of about 75 words on each of these articles.

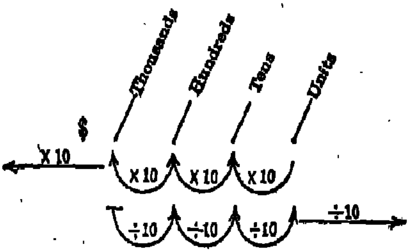
CHAPTER XXVII

TEACHING DECIMAL FRACTIONS

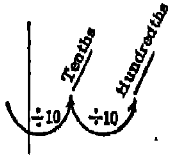
Decimal notations. The most important conception in decimal fractions is resident in the Arabic system of notation of whole numbers. When the writing and reading of all decimal forms are understood and reduced to habit, the most significant step has been taken toward the mastery of both decimals and percentage.

The rationalization. We must not assume that decimal fractions have their origin in common fractions or in the notation of our federal monetary system, although both should be used as aids in teaching decimal fractions. We are strongly of the opinion that, from the very first, decimal fractions should be presented as extensions of the system of notation of whole numbers.

The first lesson may begin by reading $1111. The teacher then elicits (1) the names of the successive places—units, tens, hundreds, thousands; (2) that each place increases tenfold in going from right to left; and (3) that each place decreases tenfold in going from left to right. The blackboard work at this point presents the following summary:



The problem is now put: "Where would I write a tenth of a dollar?" Since this is one-tenth of the unit, it obviously belongs in the first place to the right of the dividing line or in the dimes' place. So, too, a tenth of a dime is put into second place to the right of the line or into the cents' or hundredths' place. The following addition is now made to the preceding board work:



For the first few days the children write decimals of two places, using a line to separate the whole numbers from the fractions. They are then told that it is customary to use a point instead of a line. Children are thus helped to visualize the decimal points as part of a vertical line. This image, it is hoped, will lead them to keep decimal points directly under one another. When sufficient skill is developed in the notation and numeration of decimals of two places, the third place, thousandths, is taught on the principle suggested above. As the scope of the work increases, the new decimal places are added.

The gradation. The zero and the placing of the decimal point are the two most difficult elements in the notation of decimal fractions. The following gradation is therefore suggested.

1. Reading of 4 and 5 place whole numbers: 54369 = fifty-four thousand three hundred sixty-nine.	1. The object is to emphasize the fact that "and" does not enter into the reading of whole numbers. *It is incorrect* to say "three hundred *and* sixty nine."
2. 78.5 = seventy eight *and* five tenths.	2. The decimal idea has been taught; *and* indicates the decimal point.
3. 96.52 = ninety six *and* fifty-two hundredths.	3. The decimal idea carried to two places. The significance of *and* is reëmphasized.
4. .52; .4.	4. The decimal fraction without the contrasting whole numbers is now introduced.

5. .04; .09.	5. The zero that determines value is now taught.
6. .40; .80.	6. The zero that changes form but not the value is presented as a contrast.
7. 37.584; 243.504; .872.	7. The 3-place decimal fraction.
8. 7.004; .007; 7.04; .04.	8. The zeros that determine value are presented in forms already made familiar.
9. 7.40; 7.400; .400; .40; .4.	9. The zeros that do not determine value.
10. .4, .04, .004; .009, .09, .9, 9.	10. The comparison of decimal fractions that reveals the influence of moving the decimal point one or more places right or left.

The above steps, with increasing decimal places, supply material for work in decimal notation of wider scope.

Graded practice in reading, writing and interpreting decimal fractions. In writing decimals children should be taught (1) to visualize each number; (2) to recognize instantly the number of places required for a given decimal; (3) to note the number of places occupied by the number as given; (4) to decide on the number of places to be added; (5) to write the figures continuously, left to right. Thus, when the number *five thousand sixty four ten-thousandths* is given, the children should: (1) visualize 5064; (2) think "ten-thousandths require four places"; (3) think, "the given number has 4 places"; (4) think, "no extra places are needed"; (5) write a decimal point, a 5, a 0, a 6 and a 4, or .5064. To write *five hundred six ten-thousandths*, the child should be trained (1) to visualize 506; (2) to think "four places are required"; (3) to think "three places are given"; (4) to think "one place is needed"; (5) to write, left to right, decimal point, a 0, a 5, a 0, a 6, or .0506. These steps, if followed closely and understandingly for a few days, become a habit and develop speed as well as accuracy.

Every day must bring its variety of drill in decimal notation, which must gradually include the following:

1. Teacher calls the position of a decimal place, children give its name.
2. Teacher calls the name of a decimal place, the children give its position.
3. Children read decimals from books, mimeographed sheet, or charts.
4. Children write decimals that are spelled out, for example, five and seventy-two thousandths.
5. Children write decimals dictated to them.
6. Children arrange decimals so that similar places are one under the other. Thus, arrange the following correctly: .72; .074; 5.0003; .9. Pupils need practice in putting tenths under tenths, etc.
7. Miscellaneous drills.
   a. Children change form of decimals without changing their values, for example,
   Write $^{52}/_{100}$ decimally as hundredths, as thousandths, as ten thousandths, that is, $^{52}/_{100} = .52 = .520 = .5200$.
   b. Read .52, .052, .0052.
   c. Write 62½/100 as hundredths, .62½; as thousandths, .625; as ten-thousandths, .6250; as tenths, .6¼, etc. Write ⅓ as tenths, .3⅓; as hundredths, .33⅓; as ten-thousandths, .3333⅓; etc.
8. Estimating values of decimals. Which is greater, .01 or .0089, .1 or .0875, .2 or .1807, .5 or .50? Give in dollars and cents the approximate value of \$.429, of \$.6110, or \$.5106.
9. Arranging decimals in order of value, thus, .9, .87, 1., .1, are written in the following sequence, 1., .9, .87, .1.

These drills should continue until children have attained a degree of accuracy and speed in reading, writing, and interpreting decimal fractions comparable with the skill they possess with whole numbers. Time spent ungrudgingly on decimal notation is time saved in the later stages of work in decimals and in percentage.

Addition and subtraction of decimals. Two distinct skills must be developed to insure accuracy in adding and subtracting decimals: (1) to write a column of decimals with like places one under the other, and (2) to distinguish relative values of decimal fractions and whole numbers. Children who think .98 is larger than 1. are not ready to do anything with decimals. Similarly, those who write decimals vertically and do not see the significance of the error in the column to the left should be given further practice in decimal notation. Both of these preliminary exercises are provided in the practice material for decimal notation.

54.862
5.44
986.4
50.02

It seems reasonable to permit children to employ the zero crutch, especially in subtraction of decimals. Let us assume that 1.2347 is to be subtracted from 34.02. *Form A* is generally preferred; *Form B*, condemned, because the two zeros in the minuend are a crutch. If the writing of these zeros tends to make children more accurate and, by reducing concern, more rapid in their work, the crutch may be permitted.

Form A	*Form B*
34.02	34.0200
1.2347	1.2347
32.7853	32.7853

Multiplication of decimals. *The rationalization.* Two approaches are possible: (1) through common fractions; (2) through an analysis of the decimal system of notation. Each method has its sponsors, but neither enjoys the prestige of proved superiority. Those who taught the notation of decimal fractions as a continuation of the notation of whole numbers will naturally prefer the second plan; the others, the first.

1. *Through common fractions.* Give a problem in which it is necessary to multiply .2 by .4, .2 by .04, .02 by .4, .02 by .04. Elicit that 2 and 4 are multiplied and that 8 is not the

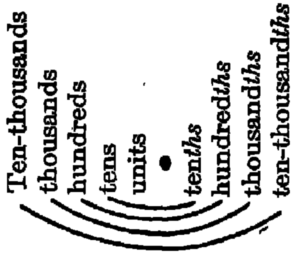
answer. Lead children to perceive that the basic problem lies in finding how to manipulate the decimal points. The blackboard work at this point shows all under column *A*.

A	*B*	*C*	*D*
.2	$\frac{2}{10} \times \frac{4}{10} = \frac{8}{100} =$	.2	1 place
.4		.4	1 place
8 ?		.08	2 places
.2	$\frac{2}{10} \times \frac{4}{100} = \frac{8}{1000} =$	.2	1 place
.04		.04	2 places
8 ?		.008	3 places
.02	$\frac{2}{100} \times \frac{4}{10} = \frac{8}{1000} =$	.02	2 places
.4		.4	1 place
8 ?		.008	3 places
.02	$\frac{2}{100} \times \frac{4}{100} = \frac{8}{10000} =$	.02	2 places
.04		.04	2 places
8 ?		.0008	4 places

We now elicit, further, that the correct answers may be found by translating these decimal fractions into common fractions. All that is shown under *B* is now worked out with the class. Each of the four 8's in the first column of work is now changed into .08, .008, .008, and .0008, respectively, as shown under *C*, all of this, however, being done by the pupils. The aim of the lesson, "to find where to place the decimal point," is then recalled. The children are guided to compare, as shown under *D*, the number of decimal places in the product with the number in both the multiplicand and the multiplier. The facts they thus acquire become the basis of the generalization for pointing off in multiplication of decimals.

2. *Analogy with whole numbers.* The second method of teaching multiplication by decimal fractions carries over the decimal relations of whole numbers to fractional forms. Through well framed questions the teacher leads the class to see the parallel of places existing on both sides of the units'

place, and puts the following diagrammatic scheme on the blackboard:



After this correspondence is carefully noted the teacher presents examples in multiplication of whole numbers as follows: 10×10=? 10×100=? 100×10=? The questions take the following form:

A	*B*
If 10× 10=100	then .1 ×.1 =?
If 10×100=1000	then .1 ×.01=?
If 100× 10=1000	then .01×.1 =?

Following the parallel that was pointed out before, the pupils readily give the answers .01, .001, and .001, respectively. The teacher then asks: "How many zeros, in Group *A*, are always contained in the answer?" Since this form of multiplication is very familiar to children in this grade there is no difficulty in obtaining from them, "As many ciphers as there are in both multiplicand and multiplier." The lesson is concluded when the teacher's question, "How many decimal places will there be in the product?" brings the answer, "As many places as in both multiplicand and multiplier."

Gradation and drills. No fine gradation is necessary in planning the lesson on the multiplication by decimals. The first examples employ numbers of the type of .7×.4, .05×.12; that is, decimal fractions without whole numbers. The second type includes mixed numbers like 7.2×1.2, 7.12×1.02, 7.12×.102, and finally whole numbers without fractions—72×.03, 12.04×16, 9×.009. Any more carefully planned gradation is unnecessary.

Provision must be made for rapid oral drills, which seek to reduce the process of pointing-off to habit. Numerous exercises like the following must be given daily in two-minute periods:

.1 × .1
.1 × .01
.01 × .1
1 × .01
.01 × .001
.01 × 1.
.1 × 1.
.01 × 1.1
.01 × .11

1.2 × .9
1.2 × 9.
.12 × 9.
.12 × .9
1.2 × .09
12. × .9
12. × .09
.12 × .09
.012 × .009

After the decimal equivalents of the common business fractions are learned, children should be led to perceive that in examples like 712×$\frac{3}{5}$, the decimal multiplier is simpler. In the first case, 712 is multiplied by 3 and the product is divided by 5; in the decimal form, 712 is multiplied by 6 and one place is pointed off. So, too, 712×1$\frac{1}{5}$ involves 712×6÷5, but 712×1.2 necessitates multiplication by 12, and proper placing of the decimal point. The decimal form of multiplying 712 by $\frac{2}{25}$ requires merely the multiplication of 712 by .08, but in the common fractional form 712 is multiplied by 2 and the product divided by 25. Regular and frequent drills, by devices already explained, develop the required skill in multiplication by decimals and thus facilitates the learning of percentage.

Division by decimals. *The rationalization.* Division in which the divisor is a decimal may be performed in a variety of ways. The decimals may be reduced to common fractions and the process carried out by inverting the divisor and then proceeding as in multiplication. If 54.62 is to be divided by .002, the example may be changed to

$$\frac{5462}{100} \div \frac{2}{1000}, \quad \text{or} \quad \frac{\overset{2731}{\cancel{5462}}}{\cancel{100}} \times \frac{\overset{10}{\cancel{1000}}}{\cancel{2}} = 27\,310$$

But it is obvious that dividing by the decimal .002 is far simpler. For practical reasons, if for no others, children must learn to divide by a decimal fraction as well as by a common fraction.

Two practices are commonly taught for division by decimals. The one known as the older method reverses the process of multiplication. By questions and examples we elicit—

1. The dividend = the divisor × the quotient
2. The product = the multiplicand × the multiplier
3. The number of decimal places in the product = the sum of the decimal places in multiplicand and in the multiplier
4. The number of places in the quotient = the number of places in the dividend minus the number of places in the divisor

The class learns quickly enough that in dividing by a decimal the decimal point in the dividend is moved to the right as many places as there are decimal places in the divisor. The decimal point in the quotient is directly above this newly placed decimal point in the dividend.

A second practice in dividing by a decimal is to multiply the divisor and the dividend by 10, 100, or 1000, etc., as may be necessary, to make the divisor a whole number. The decimal point in the quotient is directly over the new decimal point of the dividend. To teach the second practice, assign the example \$24.36 ÷ 6. No difficulty is experienced by the children. As was explained in an earlier chapter, 6 is now changed to .6 and the class does not know how to proceed. The children soon see that if the decimal-fraction divisor, .6, were changed to 6., they would then be on familiar ground. They now know the aim of the lesson. The teacher then asks the children to write $.6\overline{)25.44}$ in fractional form, and $\frac{25.44}{.6}$ is the result. The next step is to lead children to perceive: (1) that changing the denominator from .6 to 6. multiplies it by 10; (2) that this may be done provided the numerator is also multiplied by 10. The blackboard work now shows:

$$.6\overline{)25.44} = \frac{25.44}{.6} = \frac{25.44}{.6} \times \frac{10}{10} = \frac{25_{\times}4.4}{_{\times}6.} = 6\overline{)2.544}$$

A few more examples are analyzed and solved in this manner, and the children are ready to conclude that the divisor is turned into a whole number by multiplying it by 10, 100 or 1000, and that the dividend is then multiplied by the same number. Care must be taken not to have children think or say that the decimal point is moved one place to the right in both divisor and dividend. The pupil must always think and say that he is multiplying both numerator and denominator, dividend and divisor, by the same number.

The relative merits of these two plans of rationalizing the division by a decimal have been the subject of much discussion and investigation. Monroe found that the same children often use different devices for placing the decimal point, depending upon the circumstances presented by the varying examples.[1] Experimental studies seem to favor the second method, *i.e.*, multiplying both dividend and divisor by that number which will turn the divisor into a whole number.[2] The experimental data are impressive but not conclusive.

Gradation and drill. The teaching of division by decimals requires careful analysis of successive difficulties and the formulation of a full series of graded steps. This gradation is given in Chapter VII and therefore is not repeated here.

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[1] W. S. Monroe, "The Ability to Place the Decimal Point in Division," *Elementary School Journal*, Vol. 18, December, 1917, pp. 287-293.

[2] J. A. Drushel, "A Study of the Amount of Arithmetic at the Command of High-School Graduates Who Have Had No Arithmetic in Their High-School Course," *Elementary School Journal*, Vol. 17, May, 1917, pp. 657-661.

It remains to point out that daily two- or three-minute drills in rapid oral work are essential for proper habituation. Examples of the following type must be used until the decimal point is placed with absolute accuracy from force of habit.

$.8 \div 2$
$.08 \div 2$
$.008 \div 2$
$2.08 \div 2$

$8 \div 2$
$8 \div .2$
$8 \div .02$
$8 \div .002$

$.8 \div .2$
$.08 \div .02$
$.008 \div .002$
$.8 \div .02$

$.8 \div .002$
$.08 \div .2$
$.008 \div .2$
$.008 \div .02$

When division by decimals is properly understood and mechanized, the underlying principle should be applied to division in which the divsor is 1200, 800, or 2700—numbers with terminal zeros. This can be explained on the same basis as that used in decimal divisors, namely,

$$246.48 \div 800 = \frac{246.48}{800} \div \frac{100}{100} = \frac{2.46_{\times}48}{8.00_{\times}} = \begin{array}{r} .3081 \\ 8\overline{)2.4648} \end{array}$$

No great difficulty is experienced in leading the class to generalize—"divide divisor and dividend by the same number, the number necessary to free the divisor of its confusing final ciphers."

Reduction to decimal fractions. *Importance.* Reduction of decimal fractions to common fractions presents no teaching problem and will therefore receive no further attention. Reduction to decimal fractions is an important operation because of the frequent need of finding what per cent one quantity is of another. In classroom activities the teacher will find many instances requiring the reduction of common fractions to decimal forms.

Gradation and rationalization. For the purpose of reduction to decimals, common fractions may be divided into the three following groups: Group 1, those whose denominators are two, four, and ten, ½, ¼, ¾, ⅒, ³⁄₁₀, etc. Group 2, those that can be reduced by inspection, ⅛, ⅕, ⅗, ⅓, ⅙. Group 3, those that require paper and pencil operations, $\frac{5}{6} = 5 \div 6 = 6\overline{)5}$.

Group 1 is learned very quickly. Children in grades in which decimals are taught know our monetary system. Hence

¼=25 cents or $\frac{25}{100}$, ¾=$\frac{75}{100}$, ½=$\frac{5}{10}$ or $\frac{50}{100}$, etc. In each case the common fraction is written in decimal form, ¾=$\frac{75}{100}$=.75.

Group 2 requires a little more planning along similar lines. Encourage the child to think of these fractions as parts of a dollar. $\$\frac{1}{5}$ becomes 20 cents or $\$\frac{20}{100}$, expressed as \$.20 or \$.2; $\$\frac{1}{8}=12\frac{1}{2}$ cents or \$12½/100 or $\$.12\frac{1}{2}$; $\$\frac{1}{3}=33\frac{1}{3}$ cents or $\$.33\frac{1}{3}$. Another aid is to change the common fraction to another common fraction having a denominator of 10, 100, or 1000, thus:

7/125 = ?/1000
31/50 = ?/100
5/8 = ?/100

To reduce $\frac{7}{125}$ to thousandths and $\frac{31}{50}$ and ⅝ to hundredths, both terms of the first fraction must be multiplied by 8, of the second fraction by 2, and of the third fraction by 12½. The board work is now increased as follows:

$$\begin{array}{lll}
\frac{7}{125}=\frac{?}{1000}; & 1000 \div 125 = 8; & \frac{7 \times 8}{125 \times 8}=\frac{56}{1000}=.056 \\
\frac{31}{50}=\frac{?}{100}; & 100 \div 50 = 2; & \frac{31 \times 2}{50 \times 2}=\frac{62}{100}=.62 \\
\frac{5}{8}=\frac{?}{100}; & 100 \div 8 = 12\frac{1}{2}; & \frac{5 \times 12\frac{1}{2}}{8 \times 12\frac{1}{2}}=\frac{62\frac{1}{2}}{100}=.62\frac{1}{2}
\end{array}$$

It is wise, as each common business fraction is reduced to its decimal equivalent, to require the memorization of both forms. On one day the child should memorize ½, ¼, and ¾; on the second ⅛; on the third, ⅜ and ⅝; on the fourth, ⅞; on the fifth, ⅓ and ⅔. In this way useful arithmetic information is acquired without appreciable effort.

Group 3 includes fractions like ⅚ and $\frac{5}{16}$, and the less common but necessary fractions like $\frac{19}{35}$ and $\frac{15}{32}$. Obviously, the methods suggested above do not apply. Since the reduction of such fractions constitutes one of the necessary mechanical processes in arithmetic, it should be taught with a view towards habituation and therefore with a minimum of rational appeal. The mode of instruction is summarized below:

Reduce 5/16 to a decimal.

$$\tfrac{5}{16} = 5 \div 16 = 16\overline{)5.}\quad = 16\overline{)5.000}$$

$$\begin{array}{r}
5\quad .31\tfrac{1}{4} \\
\overline{16})\overline{5.000} \\
\underline{48} \\
20 \\
\underline{16} \\
4
\end{array}$$

As the child becomes more familiar with the process, he discards one or two steps in this series of equations. No deeper rationalization is necessary or even possible. The child who knows that every fraction is equivalent to numerator divided by denominator finds in the equations $\tfrac{5}{16} = 5 \div 16 = 16\overline{)5}$ an adequate explanation. All available time should be spent on habituating this process of reduction.

Common errors in decimals and the corrective measures. In our discussion of decimals we indicated the common errors that children make and the remedial drills designed to eliminate them. In listing them here we are merely reviewing the essentials teaching problems in decimals.

The general causes of the most common errors in decimals, aside from limited intelligence of the learner, are, in the main, three:

1. Inadequate mastery of the skills in the fundamental operations of whole numbers and of the principles of our system of notation.
2. Inadequate understanding of decimal fractions.
3. Failure to retain the decimal point throughout an arithmetical process.

The remedial measures lie in effective teaching: genuine motivation, careful gradation, adequate rationalization, diagnostic testing, and sufficient differentiated and individualized practice material.

Errors and Their Causes	Corrective Measures
In Addition	
1. Arranging decimal addends incorrectly in vertical column, thus: 5.04 1.6 160.8	1. a. Constant reminder, orally or by chart, "point under point." b. Practice in arranging, in vertical columns, decimals that are assigned horizontally. c. Testing the column of decimal points with ruler or card to see whether they are one under the other, before beginning to add or subtract. d. Encourage children to insert 0 in blank places: 3.072 1.500 3.090
2. Failure to retain the decimal point in the answer or to place it correctly, thus: 54.08 1.41 7.06 6225 or 625.5	2. The devices listed above under 1, *a*, *b*, and *c*, tend to correct this fault of operation. Practice in addition of the following types is also corrective, for example: 7.04 + .8; 42.5 + 8; 42.5 + .4.
In Subtraction	
1. Failure to think *zero*, when necessary, in the minuend, thus: 5 − 2.45 = 5 2.45 3.45	1. Insist that the child make the number of places in minuend and subtrahend equal, thus: 5.00 2.45 2.55
2. Changing minuend to subtrahend because the subtrahend has more places; thus, 5 − 2.45 becomes 2.45 5 2.40	2. a. Practice in estimating values of decimals and whole numbers. b. Continued practice in reading and writing of decimals and in arranging a series of decimals in increasing or decreasing magnitude.

ERRORS AND THEIR CAUSES	CORRECTIVE MEASURES
	c. Continue practice in changing the form but not the value of a decimal; in changing the value with a minimum change in form, for example, 8.5; 85.; .85; .085; 8.05; 805; .805; .52; .520; .5200.
	d. Use dollars and cents and limit decimals to two places. Insist on an interpretation of the numbers before permitting children to perform the indicated operation.
In Multiplication	
1. Misplacing the decimal point; for example, 5.4 × .02 = 1.08 2. Pointing-off in the answer by counting from left to right, thus, .54 × .2 = 108. 3. Failure to prefix zeros in the answer, for example, .52 × .02 = .104	1-2-3. These errors are due to inadequate understanding of decimal notation and of the rule for pointing-off. Through a diagnostic test the specific errors may be discovered. If the group making any one of these errors is large, the remedy is comparatively simple: teach what is not understood and supply drill suitable in form and quantity. If only a few make any one of these errors, arrange for individual help. Give daily practice of the following type: indicate the correct place of the decimal point in the following: 24 × 3 = 72 2.4 × .3 = 72 .24 × 3 = 72 24 × .03 = 72 1.36 × 12 = 1632 136 × .12 = 1632 13.6 × .12 = 1632 136 × 1.2 = 1632

Errors and Their Causes	Corrective Measures
In Division	
1. Incorrect placing of the decimal point. *a* .136 1.2)16.32, as many places in the quotient as in the dividend and the divisor; a carry-over from multiplication. *b* 1.36 1.2)16.32, a carry-over from addition, "point under point." *c* 136 1.2)16.32, failure to insert the decimal point in the quotient. *d* Failure to annex zeros to the dividend, 9. 1‸32.)1188. 1188	1. These four types of errors reveal limited understanding of decimal notation and of the process of division by a decimal. Reteaching is essential but great care must be taken to grade the difficulties in a progressive order. Whether the class or only selected pupils should be subjected to the reteaching must be decided by the teacher. Daily drills of the following types are helpful: *a.* .5 × 10, × 100, ÷ 10, ÷ 100 7.5 × 10, × 100, ÷ 10, ÷ 100 *b.* Which are correct? Why? 4.5 ÷ .5 = 9, .9, .09, .009 .45 ÷ .5 = 9, .9, .09, .009, etc. *c.* Place the number marked *answer* in the quotient place and then put the decimal point where it belongs: .06)54.36 answer, 906 .007)442.4 answer, 632 7)44.24 answer, 632 etc.
2. Errors due to inadequate mastery of the process of division in whole numbers.	2. Learning arithmetic consists of mastering a set of progressive habits. If children do not have primary skills, they must set aside advanced techniques until the earlier ones are acquired. Hence, division by decimals must wait until children learn the process of division by whole numbers.

PROBLEMS FOR STUDY AND DISCUSSION

1. A pupil cannot arrange the following decimals in order of magnitude: .03; .0083; .30; .030; 3.00. What are the causes of this pupil's deficiency?

2. List all the decimal forms that pupils in the junior high schools find difficult. How would you help such pupils overcome these difficulties?

3. Analyze one of the published diagnostic tests in decimals. What weaknesses does the test seek to reveal? Compare these with the common errors listed in this chapter. What differences do you note?

4. Select from this chapter the six teaching suggestions that you consider most important. Tell why each is important.

5. Devise a test in decimal notation. What specific facts and skills are you testing? What types of questions do you plan to employ? What precautions will you take to achieve a well balanced test? Find such a test in a textbook in arithmetic. How does your effort compare with the test in the book?

6. Write a paragraph on each of the following:
   (*a*) The Roman numerals are decimal in nature.
   (*b*) The invention or the discovery of the zero in mathematics is of overwhelming importance.
   (*c*) One of the earliest known decimal systems.

CHAPTER XXVIII

TEACHING PERCENTAGE AND ITS APPLICATIONS

Introducing the percentage concept. Every teacher knows that in percentage a new name is applied to a familiar arithmetical form. But in teaching percentage the topic is too frequently presented as remotely related to what the children already know. The best initial lessons on percentage are almost incidental. The first lesson may develop along the following lines:

Ask the children to solve: Goods cost $500, were sold at a profit of .05, of .06, of 8/100 of the cost; find the gain. Elicit known mode of solution: multiply and point off two places.

The teacher now tells the class that in business the word *hundredth* is not used; neither is it used in class, for no one speaks of having received seventy hundredths in arithmetic. Elicit the term *per cent.* Post the following conspicuously:

Per cent means *hundredth.*

Per cent of means *hundredth times*, for example, *5% of $250 means $250 × .05.*

Ask the children to solve the following: Goods costing $600 were sold at a profit of 5 hundredths or 5 per cent, .07 or 7%, 8/100 or 8%; find the gain. Show the symbol, %.

Assign problems in known business applications, using the form, 7½%, 9%, 12%, without the expression hundredths.

The children should not be aware that a new division of their work has been introduced, nor should the Latin *per* (by, or out of) and *centum* (hundred) be brought into the lesson. This etymological story tends to confuse a concept that is comparatively simple.

Developing the percentage concept. A series of well planned lessons will do much to forestall certain difficulties that arise in learning computation by percentage. The errors made by children in percentage are due chiefly to lack of knowledge of decimals, common fractions, and the notation of whole numbers. A test on the fundamental principles of these branches of arithmetic will reveal the weaknesses that must be strengthened in review lessons. The following exercises have preparatory value:

1. Change two-place decimals with or without a fraction to per cent forms.
   .12 = 12 hundredths = 12 per cent = 12% (teach symbol, %)
   .12½ = 12½ " = 12½ " " = 12½%
2. Change one-place decimals to per cent forms.
   .7 or .9 to 70% or 90%
3. Change common fractions to per cent forms.
   $\frac{3}{25} = \frac{12}{100} = .12 = 12\%$
4. Change indeterminate decimal fractions to per cent forms.
   ⅓ = .33⅓ = 33⅓%
   ⅙ = .16⅔ = 16⅔%
   ⅑ = .11⅑ = 11⅑%
5. Express 100 per cent of any quantity.
   $\frac{4}{4} = 1. = 100\%$
   $\frac{5}{5} = \frac{6}{6} = \frac{8}{8} = \frac{12}{12} = 1. = 100\%$
6. Express more than 100 per cent of any quantity (whole and mixed numbers)
   2 = 2.00 = 200%
   $\frac{5}{4}$ = 1¼ = 1.25 = 125%
   $\frac{8}{5}$ = 1⅗ = 1.60 = 160%
   $\frac{9}{4}$ = 2¼ = 2.25 = 225%
7. Express fractions of a per cent (three and four decimal places)
   .04¼ = 4¼ hundredths = 4¼%
   .06⅓ = 6⅓ hundredths = 6⅓%
   .08½ = 8½ hundredths = 8½%
   .04¼ = .0425 = 4¼ hundredths = 4¼% = 4.25%
   .06¾ = .0675 = 6¾ hundredths = 6¾% = 6.75%
   5½% = .05½ or .055
   8¼% = .08¼ or .0825
   12½% = .12½ or .125
   9¾% = .09¾ or .0975
   7⅛% = .07⅛ or .07125
8. Express less than 1 per cent
   ½% = ½ hundredth = .00½ = .005
   ¾% = ¾ hundredth = .00¾ = .0075
   .00125 = ⅛%
   .0033⅓ = ⅓%

The final test of reasonable mastery of the concept per cent is absolute correctness in the following:

.01 = 1%
.1 = 10%
1. = 100%

.30 = 30%
.3 = 30%
3. = 300%

.0125 = 1¼%
1.75 = 175%
.005 = ½%

5% = .05
.5% = .005
.50% = .5
500% = 5

A well planned test[1] revealed that less than 2 per cent of the children failed in the following: change .55, .06, 1.24, 1.13, 1.63 to equivalent per cent forms. But from 40 to 59 per cent failed to change the following to per cent forms: .0325, .065, .0475, .4. The query, 9/100 of N=   %, brought many incorrect answers in the seventh grade; the most common of these are 90% and 900%. The children did not see that 9/100 is a small part of N and that 90% and 900% are, therefore, absurd answers. They were carried away by the zeros in 9/100. These common errors give force to our statement that too much attention can hardly be given to decimal notation for most errors in percentage may be ascribed to imperfect learning of decimal fractions.

Six difficulties in percentage. Extensive testing reveals six common errors in percentage. "Gifted children do not make one set of errors and slow pupils another; both types of pupils make the same errors," in about the same relative proportion.[2] These six types of errors indicate, therefore, where most practice must be concentrated.

1. Less than 1%; examples: .006, 5/300, ¼%, ¼ of 1%.
2. The recognition of the many equivalent forms;
   5%, 5/100 of N, .05, 5% × N.
3. The combination of types 1 and 2;
   .065 = .06½ = 6½% = 6.5%
4. One-place decimals reduced to per cent form.
   .4 as 40% not 4%.

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[1] Arthur Edwards, "A Study of Errors in Percentage," in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education, Part II (Public School Publishing Co., 1930), Ch. xii. A very valuable study.

[2] *Ibid.*, p. 640.

5. Whole numbers in per cent form.

$$5 \times N = 500\% \text{ of } N;\ 1 = 100\%.$$

6. Mixed numbers in per cent forms.

$$\frac{7}{4} = 1\frac{3}{4} = 1.75 = 175\%$$

$$124\% = 1.24 = 1\frac{24}{100} = 1\frac{6}{25} = \frac{31}{25}$$

Per cent equivalents of the common business fractions. In the discussion of reduction of common fractions to decimals we stressed the importance of having children memorize the decimal equivalents of common fractions which occur in business. We must now add the per cent form to the decimal form. The list of these fractions includes $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{10}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$. Such fractions as $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{12}$, $\frac{1}{15}$, and $\frac{1}{16}$ are not common enough to be taught in this way. Not all the common fractions will be reduced to per cent equivalents on any one day. It is wise, therefore, to reduce one or two of these fractions daily and to memorize them upon the completion of the computation. Children soon see the value of this memorized information. They are quick to perceive the advantages of $\frac{3}{4} \times \$1.68$ over 75% of 1.68 or $1.68 \times .75$. Learning per cent forms of common business fractions may be greatly facilitated by motivating examples like the preceding one, and by exhibiting a convenient chart showing common fractions and their per cent equivalents.

The cases of percentage. The second significant step in learning percentage is the mastery of the cases. Concretely expressed, these cases are: 25% of 120; 30 is what per cent of 120; if 25% of a number is 30, what is the number? Most of the teaching problems arise in the failure to realize that each case in percentage involves no new relationship of numbers; that these cases occurred in common fractions. In his work in fractions the child solved problems like these: $\frac{1}{5}$ of 25; what part of 25 is 5; how many marbles did he have originally if, in losing $\frac{1}{5}$ of them, he lost 5. In teaching any of the cases of percentage, therefore, problems in the corresponding case in common fractions should be given first and then the per cent form should be substituted.

A fuller summary of the cases may make clear the relationships involved and the small difference between fractional and percentage forms:

Fractional Form	*Percentage Form*
Case I. $150 must be raised for a poor family; ⅗ of this sum is collected the first day; how much of the necessary sum has been collected?	Case I. $250 must be raised for a poor family; 45% of this sum is collected the first day; how much of the necessary sum has been collected?
Case II. $40 was collected to help a poor family; if $150 is needed, what part of the money is already assured?	Case II. $60.00 was collected to help a poor family; if $240 is needed, what per cent of the money is already assured?
Case III. (*a*) A merchant who sells to gain ⅙ of his cost finds that he gained $15. What did he pay for his material?	Case III. (*a*) A merchant who sells to gain 12% of the cost finds that he gained $36. What did he pay for his goods?
(*b*) A train that completed ⅙ of its distance has 120 miles still to run. What is the length of the entire trip?	(*b*) A train completed 16% of its trip the first hour. It has 168 miles still to run. How long is the trip?
Goods sold for $24.00 involved a loss of ¼ on my investment. What was the investment?	Goods sold for $25.80 involved a loss of 14% on my investment. What was the investment?
(*c*) Goods sold for $25.50 brought a profit of ⅙. What did they cost?	(*c*) Goods sold for $14.44 a gross, brought a profit of 12%; find the cost per gross.

We list here all the cases of percentage or fractions. Many supervisors and teachers are strongly opposed to teaching the indirect cases, Cases II and III, with all their modifications, because of their limited social use. Others are just as firmly convinced that all these cases should be taught. They argue that the indirect cases (Cases II and III) give meaning to the direct case (Case I) and make clear the full relationship existing among base, rate, and percentage. Further, they refute the statement that the indirect cases arise only on rare occasions and point to such applications as the following:

1. I paid \$3.50 a yard for goods. How shall I mark them so that I can allow the usual trade discount of 12½%.
2. Goods in a show case are marked 48 cents apiece. A sign reads, "All articles have been reduced 20%." What was the first price?
3. What is the approximate income of the Joneses who pay an annual rental of \$1250. A family usually spends 20% of its income on rent.
4. A printer estimates that the typewritten sheets given him by the author will make up 420 printed pages. The author says that this is about 75% of his book. How many pages will the book probably contain?
5. A report says that in 27 days 45% of a road job has been completed. About how long will it take to complete the whole road?

To be sure, these indirect cases are useful, but relatively they have less frequent practical application than the direct case. Case II arises more frequently in real situations than Case III. It may be well to take a middle ground in this dispute: teach the indirect cases to the superior pupils only; the others will probably not be able to learn them well enough to insure reliable application.

The business applications of percentage. The most practical phase of percentage is its application to everyday business situations—profit and loss, trade discount, discounting notes, commission and brokerage, taxes, and interest. We must suggest general teaching principles that are applicable to all of them.

All applications of percentage express the same arithmetical relations. In teaching each of the business applications teachers must lead their pupils to recognize the repetition of the cases of percentage. Mathematically considered, interest, commission, and trade discount involve the same operations. In the following problems the arithmetical relations among the numbers are identical despite any variation in the business situations:

1. $575 left in a savings bank for 1 year at 4% will earn what interest? Interest is paid semi-annually.
2. An agent sold $2454 worth of goods in 1 month. What was his commission at the rate of 5%?
3. Goods marked $24.40 are sold to retailers at a discount of 15%. What discount may a retail merchant expect?

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1. $575 left in a bank for 6 months earned $11.50 interest. What was the rate of interest? The bank pays interest semi-annually.
2. An agent sold $2554 worth of goods and charged $127.70 for his services. What was his rate of commission?
3. Goods listed in the catalog for $28 a dozen are sold to retailers for $3.20 less. What rate of discount is allowed?

The mathematical identity that exists among all of the business applications of percentage is clearly shown in the tables on page 404. The three elements of percentage, base, rate, and percentage, have their three corresponding terms or names in each of the applications; the mode of solving each case must therefore be the same. It is a profitable exercise to translate each of the symbolic representations and formulas in the table into a real problem. It is obvious that this table serves as a summary for the teacher and is not intended for the children.

Every new application to be introduced by a language lesson. The lesson that introduces a new application of percentage must be essentially a language lesson. Since all forms of business applications of percentage are mathematically identical, the instructional problem resolves itself into an endeavor to make clear the meaning of the new business terms. In teaching the general subject of interest the terms, *interest, principal, rate, note,* and *discount* must be made clear. So, too, in insurance, the terms, *face of policy, premium, life insurance, fire* or *accident insurance* must be explained thoroughly. The child who knows the cases of percentage and who has a clear comprehension of the meaning of discount, catalog price, list price, and net price can solve almost any reasonable problem in commercial discount. Too much attention can hardly be given to the language phase in this branch of arithmetic.

Motorize the business application of percentage. To stimulate vivid imagery of an actual business procedure, dramatize it. In teaching stocks and bonds a company should be organized in class, specimen shares should be written out, shares should be sold at par, at a premium, and at a discount, and dividends should be declared. Through such motorization a corporate organization becomes real, its *modus operandi* is understood, and its business advantages clearly perceived. In teaching insurance sample policies should be shown and filled out to illustrate the most common forms of insurance. So, too, in teaching interest and elementary banking, children must fill out checks, check stubs, and deposit slips; they must endorse checks and make them payable to one another, compute interest, make pass-book entries—in a word, perform every necessary business practice. No elaborate equipment is necessary to make each business procedure real and vital.

Children must be given reasons for various business practices. To understand the terms and the practices of business is not enough. Children should be taught the reasons for the practices of business. Many pupils who had no difficulty in solving any of the textbook examples in interest had no understanding whatever of the reason for paying interest to depositors. Almost all the pupils in the class believed that money deposited in a bank is securely locked in large safes that defy burglars. When these children were asked whether it would not be fair for the depositor to pay the bank for this service, a great many thought that it would.

Amusing information awaits the teacher who has not asked pupils why insurance companies pay the full face of the

Business Applications	Percentage	Profit and Loss	Trade Discount	Insurance	Interest	Commission and Brokerage	Etc.
Elements	Base Rate Percent.	Cost or s. p. Rate of g. or l. Money g. or l.	Listed price Rate of dis. Discount	Policy Rate of ins. Premium	Principal Rate of int. Interest	Goods bought or sold Rate of com. or b'kge. Com. or b'kge.	etc.
Case I	$P = B \times R$	Money g. or l. = c. or s. p. × % g. or l.	Dis. = listed price × % discount	Pre. = pol. × Rate of ins.	Int. = prin. × % int.	Com. or b'kge. = purchase or sales × % com. or % b'kge.	etc.
Case II	$R = \frac{P}{B}$	% g. or l. = $\frac{\text{money g. or l.}}{\text{c.}}$ % g. or l. = $\frac{\text{money g. or l.}}{\text{s. p.}}$	% dis. = $\frac{\text{dis.}}{\text{list price}}$	Rate of ins. = $\frac{\text{pre.}}{\text{pol.}}$	% Int. = $\frac{\text{int.}}{\text{prin.}}$	% Com. or b'kge. = $\frac{\text{com. or b'kge.}}{\text{purchase or sales}}$	etc.
Case III	$B = \frac{P}{R}$	C = $\frac{\text{money g. or l.}}{\text{\% g. or l.}}$ S. P. = $\frac{\text{money g. or l.}}{\text{\% g. or l.}}$	List price = $\frac{\text{dis.}}{\text{\% dis.}}$	Pol. = $\frac{\text{pre.}}{\text{Rate of ins.}}$	Prin. = $\frac{\text{int.}}{\text{\% int.}}$	Sales or purchase = $\frac{\text{com. or b'kge.}}{\text{\% com. or \% b'kge.}}$	etc.

policy in case of death or fire. The following cases were presented to fourteen-year-old pupils:

> B, who pays a small premium of $15.50 annually for four years, receives $3000 from the fire insurance company when his household goods are destroyed. A, who paid only $150 a year for four years, died; his family received $5000. Why did these companies pay so much to people from whom they collected so little?

The answers that the children gave seemed to show that, in their minds, an insurance company is a philanthropic institution composed of individuals of fabulous wealth and boundless compassion. Evidently no attempt was made to acquaint the pupils with the law of chance. Cursory investigation will show that children do not know why agents are paid commission rather than a fixed salary, why goods are listed at one price and sold at a discount, why double discounts, 20 per cent and 10 per cent, are given rather than one large single discount, why banks discount notes, why stocks worth $100 one year are sold at a discount the following year, why people who are older pay a higher rate of insurance, why overhead charges are included in a computation of cost, why certain houses are given lower rates of insurance than others—in a word, the reasons for the common practices of business. This information is far more significant than mere arithmetic. Even this elementary study of business practices gives the child more than a glimpse into the social studies. For many children it is an introduction to our economic structure.

Actual computation may be omitted. In teaching certain of the business applications highly socialized recitations with no actual computation are very desirable. In insurance, for example, children should be given a clear understanding of the values of insurance, the common forms, and the conditions that make possible the payment of the face of the policy to the insured or to the beneficiary. The school child who understands why duties are levied and how they are computed may well eliminate the actual computation of duties. In the past we underestimated the significance of informational arithmetic.

The mode of computation of the business world must be taught. In teaching profit and loss children should not be taught that gain or loss is always computed on the cost. In business the rate of profit is often computed on the volume of business, that is, on the selling price. In the example, "Goods sold for $12 brought a profit of $2.00; what is the rate of profit?" the formula,

$$\frac{\text{Gain}}{\text{S. P.}} = \frac{2}{12} = \frac{1}{6} = 16\tfrac{2}{3}\%, \text{ is as correct as, } \frac{\text{Gain}}{\text{Cost}} = \frac{2}{10} = \frac{1}{5} = 20\%$$

The problem must indicate on which base the rate of profit or loss is to be computed.

The method of computing double discounts is another illustration of the importance of following business practice. If discounts are 33⅓ per cent and 10 per cent, the simplest solution is to set down "66⅔ per cent of catalog price" as the new list price after the first discount; 10 per cent is now allowed on this 66⅔ per cent, or 6⅔; deduct 6⅔ from 66⅔ and the remainder, "60 per cent of the original catalog price," is the final selling price. This mode of computing is direct and superior in every way to the long process of finding the first discount, the new price, the second discount, and, then, the final price.

In finding the interest on a given sum of money for years, months, and days, common business practice follows (1) the six per cent base; (2) the method of aliquot parts; (3) the method of cancellation; (4) or the use of prepared tables. Teachers must find which of these modes is the most prevalent in the local community.

For reference purposes only, we shall illustrate each of these practices:

1. The six per cent method. Economy of time and effort results from the use of a fixed base in computing the interest regardless of the rate. The usual base is 6 per cent. Children learn that the interest on $1.00 for 1 year will be $.06; for 6 months, $.03; for 2 months or ⅙ year, $.01; for 1 month,

$.005; for 10 days or ⅙ of 2 months, $.00⅙; for 6 days, or ⅒ of 2 months, $.001. If interest is sought at 5 per cent, the result obtained by using 6 per cent is decreased by ⅙ of itself; if 4 per cent, by ⅓; if 3 per cent, by ½. Since 6 per cent is used as a basis in all examples, children become expert in finding interest by short cuts. To illustrate: Since the interest for 60 days at 6 per cent is 1 per cent, we can find the interest on any amount for 60 days at 6 per cent by moving the decimal point two places to the left.

2. Method of Aliquot Parts. By the method of aliquot parts the ratios are sought in the time rather than in the rate of interest. Some period, usually a year, is taken as a base and the additional time is computed on it.

Find the interest on $450 for 7 months and 23 days, at 6%.

Interest for 6 mos. =	13.50	(½ yr. hence ½ of $27.00)
Interest for 1 mo. =	2.25	(⅙ of int. for 6 mos.)
Interest for 20 days =	1.50	(⅔ of 2.25; 20 days = ⅔ of 1 mo.)
Interest for 3 days =	.225	(3 days = ⅒ of 30 days, hence take .1 of $2.25)
Interest for 7 mos. and 23 days is	17.475;	ans. $17.48

3. By cancellation. A convenient method of finding the interest on notes that have run a given number of days is to find the interest on the total amount involved for one year, and then to find the interest for the fractional part of a year that actually elapsed. This may be illustrated in the following problem:

"A note dated March 18, 1934, for $180 with interest at 6 per cent was paid May 5, 1934. Find the interest."

Note ran 13 days in March, 30 in April and 5 in May, a total of 48 days. Express 48 days as 48/360 of a year. The interest on $180 for one year would be $180 × .06, hence for 48 days, the interest may be expressed in the following form that encourages cancellation:

$$\text{Int.} = \frac{\$\cancel{180} \times \overset{.03}{\cancel{.06}} \times 48}{\underset{2}{\cancel{360}}} = \$1.44 \text{ answer.}$$

4. Other methods of computation. Interest may also be computed by a formula after the process involved and the logic of the various steps have been reduced to habit. The formula is introduced at a point where it sums up in convenient form a procedure that is thoroughly comprehended. Another method of computing interest selects 12 per cent or 36 per cent as the base instead of the prevalent 6 per cent. The steps in the solution are identical with those of the 6 per cent method and the advantages of the 6 per cent base hold for these bases.

It is advisable to teach children to use interest tables in simple and in compound interest. Children must know the value of these tables and the method that was followed in developing them. All good textbooks in arithmetic have reproductions of these tables. A little practice in the use of these tables develops surprising skill.

Problems for Study and Discussion

1. Prepare a test of two parts on the content of this chapter: Part I to consist of new-type questions; Part II of ten questions calling for very short essay answers. Show that your test covers the ground fully and calls for thinking as well as for remembering. In what way did the preparation of the test help you to master this chapter?

2. Show in what specific ways control of decimals facilitates mastery of percentage.

3. "There are seven chief skills involved in working with percents." Name them. For each, indicate (*a*) what preparatory knowledge and skills the pupil must have; (*b*) how you would proceed to develop this new skill.

4. How would you help pupils who make the following errors:

$7\frac{1}{2}\% = .7\frac{1}{2}$ $\qquad$ $.005 = 5\%$
$12.5\% = 12.5$ $\qquad$ $2\frac{1}{2}\% = .2\frac{1}{2}$
$500 = 5.00\%$ $\qquad$ $80\% = 8.$

5. What experimental studies have been made on percentage? Where will you look for the data? If you were inclined to read three of these, which would you select? What would guide your choice?

CHAPTER XXIX

TEACHING MEASUREMENT

Place of denominate numbers in an elementary course. Units of measure must be included in the course in arithmetic because they are daily media of evaluation and exchange. The character of these units of measure is often an index of the degree of development of a people. These measuring units have kept pace, step by step, with the growth of commerce, industry, and science. The more primitive the people, the cruder are their units of measure; the more interrelated and specialized a society becomes, the more refined and accurate are its units of measure. Old courses of study undoubtedly prescribed an excessive amount of subject-matter in the field of denominate numbers. Discriminating selection and liberal elimination are therefore essential.

Gradation in the use of measurements. There are at least three important steps that mark graded teaching of units of measure. The first consists of exercises in *general comparison.* Every kind of object or quantity should be compared. Two books held on the palm of one hand may be compared with a number of books balanced in the other. Lines of varying length are drawn on the board and their relative lengths are judged. The distance between any two objects in the room may be similarly compared with other distances. So, too, boxes, children, vases, bottles, articles of furniture, blackboard rectangles, and what not may be used in these comparison exercises to train the senses and give a feeling for *more, less, larger, smaller, heavier, lighter, shorter* and *taller*. In the beginning the elements contrasted are large and the differences are very marked. As progress is made both the size of the objects and their differences are decidedly smaller.

In the second step, *defined* but varying units are used. A cup, a chalk box, a step, a span of the hand, a drinking glass, a spoonful, and a handful, may be the measuring media. The use of standardized or legally defined units marks the third step. Foot, quart, inch, dollar, cent, ounce, pound, or square inch are examples of these standardized units.

Omit all tables not commonly used or those necessary only in technical industries. The elementary-school course should teach those tables of weights and measures used by all citizens in the course of daily activities. Troy weight, apothecaries' weight, surveyor's measure, long ton and hundred weight, cord-foot, cord, perch, and stone measure are illustrations of tables and units that have no place in a general course in arithmetic. It seems reasonable, therefore, to limit the work of the elementary school to the following:

1. Linear Measure—inches in foot and yard, feet in yard, feet in a mile
2. Liquid Measure—gill, pint, quart, gallon
3. Dry Measure—pint, quart, peck, bushel
4. Avoirdupois Weight—ounce, pound, hundred weight and ton
5. Time—seconds, minutes, hours, days, weeks, months, leap year
6. Money—United States currency and the pound sterling, the franc, the lira; and their approximate value in terms of dollars and cents
7. Surface and Volume Measures—square inch, square foot, acre, cubic inch, cubic foot, cubic yard
8. Metric System—in the junior high school and as part of the work in science
9. Useful facts not part of a traditional table—dozen, score, gross, 16 fluid ounces make a pint, a pint of water weighs about a pound, 231 cubic inches in a gallon, 2 glassfuls make about 1 pint, the abbreviations of the various units, and such other information as is distinctly useful in local practices

All units must be taught through use. Children should learn the units of measure by measuring with them. To most children *foot* recalls the purely verbal twelve inches, and not a distance. None of these units calls up the measure it represents. Teachers know how few of the children who have used these common units for at least three years can give the approximate height of the classroom door, the width of the window, the weight of the books carried daily to and from school, or the length and width of a surface containing 120 square inches. Children in a 6 B class, when asked to approximate the height of an ordinary room, gave answers varying from 10 to 60 feet. Only about 10 per cent approximated the correct answer. When a class of college juniors was asked to estimate the height of an ordinary four-story brick building, the answers ranged from 35 to 200 feet; not more than 20 per cent gave a reasonable answer. Too often units of measure remain empty words because early teaching neglected to give an adequate perceptual setting.

In teaching a unit of measure, *use* and *sense appeal*, therefore, should be the key-notes of the process. A balance-scale, a number of weights, actual measures used with liquids, groceries and vegetables, clock dials, blocks that represent a cubic inch and a cubic foot, a spring scale—all these must constitute but a small part of the school equipment for arithmetic. Children are asked to hold a pound weight in each hand; to hold a pound weight in one hand, and then to replace it with a number of books judged to weigh a pound; to estimate the weight of various boxes containing sand, and the weight of different objects of various sizes. In each case the estimate is checked up. These exercises should be repeated for two or three minutes each day until the pound means a definite weight and recalls a certain resistance. In similar exercises the quart, the gallon, the pint, the foot, and the ounce must be taught. Children who spend two minutes each day judging various distances in the class and checking their answers will, at the end of a fortnight, show marked improvement in estimating distance. Children should know the width of an average city lot, the height of a room, of an ordinary four-story building, and of a modern skyscraper, so that 20 feet or 100 feet will call up a distance twice the height of an average room or the depth of a city lot, respectively. Two hundred and thirty-one cubic inches should be learned by making a cardboard box 7 by 11 by 3 inches, filling it with sand, and then comparing the contents of the box with that of a gallon. Unless units are learned through use, the tables based on them are verbal associations that can never function intelligently.

The complete tables in the formulation we know so well should come late in the school course and should be preceded by a knowledge of common units taken from all useful tables. The child should know quart, cent, dime, foot, hour, day, and week before knowing any one of the complete tables. From each table we select such units as are most frequently used and such as will lend themselves most readily to objective demonstration. For these reasons children learn foot before inch, quart before pint, and cent before dollar.

In teaching a table it is not advisable to present it in complete form to the class. Children who have been taught the units of any measure should be led to discover the necessary equivalents by actually measuring, and should then formulate these relations, step by step, as they progress. The old teaching technique followed the order: words, thoughts, then things; the modern procedure—things, thoughts, and then words—is not only more psychological but is also more acceptable to the pupils themselves.

Denominate numbers to be used only in real situations. The complaint that school arithmetic is too often divorced from business practice finds much support in the analyses of examples in denominate numbers. The following examples, not even remotely related to the practical needs of life, are taken from books and classroom blackboards:

Reduce 2A., 4 sq. rods, 8 sq. ft. to sq. in.
Reduce 247 856 inches to higher denominations.
Reduce 1 mile, 1 rod, 1 yard, 1 foot, 1 inch to inches.
Express 58 rods, 2 yards, 2 feet, 2 inches as a decimal part of 3 miles.

Jane bought a five-cent chocolate bar on the wrapper of which she read, "Net weight 1½ oz." How much would a pound cost at this rate? (Surely not 53 cents because larger quantities wrapped in pound or half-pound packages cost less per ounce.)

As far as possible, denominate numbers should not be taught as an independent branch of arithmetic. Units of measure must be incorporated in problem-solving. Frequent incidental inclusion of denominate numbers will afford adequate practice in the necessary units of measure as well as interesting application of fundamental operations.

THE ARITHMETICAL OPERATIONS IN DENOMINATE NUMBERS

The essential arithmetical operations in denominate numbers are (1) addition, subtraction, multiplication, and division; and (2) reduction to equivalent higher and lower denominations. The teaching problems are simple enough if care has been taken to carry out the suggestions listed above for teaching the units of measure and developing the various tables of weights and measure. Each of the processes is logical and can be deduced by the class under the guidance of the teacher after adequate motivation has been assured through a genuine initial problem.

Two common errors are made by most children in working with denominate numbers. They either fail to reduce the various denominations or they perform the operations as if they were working with abstract numbers. Let us illustrate each of these errors:

Add,	5 bu.	3 pk.	3 qt.	1 pt.	
	4	5	4	0	
	7	6	3	1	
One erroneous answer	16	14	10	2	No reduction of smaller to larger denominations.
Another erroneous answer	17	5	0	2	Child does what he would do in ordinary addition, 3 qt. + 4 qt. + 3 qt. = 10 qt., put down 0 and carry 1.

	bu.	pk.	qt.	
From	2	3	3	
Take		2	4	
One erroneous answer	2	1	1	No denominations; numbers reversed to facilitate subtraction.
Another erroneous answer	2	0	9	Child thinks 4 from 3; 4 from 13 = 9; 2 from 2 = 0. In a word, the process of ordinary subtraction is followed here.

Multiply, 2 gal. 2 qt. and 1 pt. by 5.

	gal.	qt.	pt.	
	2	2	1	
			5	
One erroneous answer	10	10	5	Child multiplies and neglects denominations and their reductions.
Another erroneous answer	11	0	5	Child multiplies as in ordinary multiplication He thinks 5 × 1 = 5; 5 × 2 = 10, write 0 and carry 1; 5 × 2 = 10, now add 1, result 11.

The fundamental operations in denominate numbers presuppose ability to reduce lower denominations to higher, that is, reduction ascending (pints to quarts and gallons), and higher denominations to lower, reduction descending, (gallons to quarts and pints). Here again, no serious teaching problem arises if children have acquired the preparatory knowledge of the units of measure.

The metric system. Many authorities insist that the metric system be made part of the course in denominate numbers because: (1) its decimal character makes almost all of its multiplications and divisions a mere matter of moving the decimal point; (2) it is so common in Europe that foreign bills and commercial paper sent to America are often computed in terms of its units; (3) it is the system employed almost exclusively in science; (4) the movement for its introduction into the United States is making headway. While there are no immediate prospects that the metric system will be legalized, the school graduate should nevertheless understand what he sees about this system in the newspapers. Although the superiority of the metric system is conceded and its uniformity and simplicity of manipulation are admitted advantages, there is a strong sentiment against abandoning our clumsy measures: first, because of tradition, and second, because of the necessity of completely changing all tools and machines from the one basis to the other. To many, the cost and the very real inconveniences involved in such a change are deciding factors.

Teaching the metric system. *Teach the three basic units.* The first step is to teach children, as objectively as any other units, the three basic measures of the metric system: the meter, 39+ inches; the liter, the contents of the cube the side of which is .1 of a meter; the gram, the weight of a cubic millimeter of distilled water. This much should be taught all children but the more refined units and their use should be reserved for the senior high school and should be incorporated in science work.

Teach the diminutive and augmentative prefixes. After the children have learned the metric units of length, capacity, and weight, and are confronted by the need of employing the metric system in their study of science, the diminutive prefixes should be evolved inductively by using our decimal system as an aid. The .1 gives the *deci*, the .01 gives the *centi* and the .001 give the *milli*. These prefixes will readily lead to .1 meter or decimeter, .01 meter or centimeter, .001 meter or millimeter.

The prefixes of increase are more difficult for children because they are Greek derivatives. These are taught deductively and with due emphasis on the American equivalents. We must make sure that the *deka* (×10), *hekto* (×100), *kilo* (×1000), and *myria* (×10,000), are thoroughly understood. The augmented measures are now formulated, and when possible, reference is made to them in science.

All work in the metric system should be given with the comparative chart in full view. These charts showing our measures and the metric equivalents can be obtained from the appropriate Federal Bureau and from many manufacturers of school equipment. At every stage of the work we must give children this sense of equivalents. Unless we constantly translate one measure into the other, the metric units will degenerate into mere names.

Mensuration

The teaching of mensuration is replete with as many teaching abuses as the teaching of denominate numbers. This branch of arithmetic is expanded beyond all rational bounds, more because the field is rich in mathematical possibilities than because of any social justification.

Mensuration should be taught for many reasons: it has a variety of practical applications in computing distances, perimeters, areas, volumes. It is used to estimate cost of quantities bought in terms of standardized units, of cementing walks and cellars, and of materials used in the creative arts. The various units of measure are constantly reviewed in these practical situations. Gradually the children acquire a large stock of useful concepts and important terms. They learn to recognize distances, quantities, volumes, and forms, and to know the meaning of base, altitude, perpendicular, rectangle, triangle, parallel—a host of words that are basic in elementary mathematics, and also in many trades.

Teaching principles governing mensuration. *Knowledge the result of actual experience.* To insure clear comprehension children should be required to perform all drawings, measurements, and constructions involved in learning any principle of mensuration. Every child must be led to formulate the rules for finding the areas of rectangles, triangles, parallelograms, and the volume of a box or room. By a process of construction and superposition of parts, the child can evolve the generalization. No rules should be given nor should the teacher's blackboard drawings be the only objective aid. Every arithmetic lesson in mensuration should also be a lesson in drawing and manual arts for each pupil.

The mode of computation must be the one used in industry. To continue teaching children the logical practices in papering, plastering, and carpeting—evolved by teachers and authors of books in arithmetic but not used in actual industry—is to widen the breach between classroom and life.

Practice in oral estimates. Much of the work in mensuration should consist of exercises which call for approximate lengths, areas, volumes, perimeters, and costs. This work, conducted orally and continued until a fair degree of accuracy is developed, is exceedingly valuable for practical needs in various business situations.

Encourage the use of short cuts. Since mensuration involves much computation, short cuts should be employed when possible. Teachers should occasionally formulate problems with numbers that encourage the use of the common short cuts. Instead of finding the cost of excavating a cellar 12×39×8 yards, select one that is 12½×39×8 yards. The children naturally multiply 12½ by 39 and then multiply the result by 8. When they are shown that 12½×39×8=12½×8×39; that 12½×8 is 100; and that 100×39 is 3900, they realize the importance of examining all numbers in search of convenient ratios before beginning actual operations.

ILLUSTRATIVE LESSONS IN MENSURATION

In the concluding section of this chapter we shall apply the principles we prescribed in teaching the most common practices in mensuration.

Drawing to scale. Practical considerations and the demands of map work in geography make necessary an understanding of the meaning of drawing to scale and the techniques involved. We give below, graded exercises designed to develop mastery of the information and the practices that are involved in reading and making drawings to scale.

1. Draw on board top view of chalk box, of circular inkwell, and of the teacher's desk. Use exact measurements. Children make similar drawings on board and on large sheets of paper.
2. Introduce need of scale: How shall we make a drawing of the floor on the blackboard? Elicit from them that a figure having the shape but not the size of the floor can be drawn.
3. Introduce idea of scale. Draw plan of floor; scale, inch on board represents one foot of the floor measurement.
4. Draw to scale a wall, the hallway, the street, the playground. In scale drawing of the classroom floor introduce the large articles of furniture and location of doors, windows, closets, etc.
5. Introduce varying scales for same object. Draw plan of the floor to scale, 1 inch = 1 foot and 1 inch = 1 yard.
6. Draw three rectangles on the board, each 4 × 3 feet. In one rectangle draw plan of the room, scale 3 inches = 1 foot; in the second, a plan of the yard, scale 1 inch = 1 foot; in the third, a plan of the block, 1 inch = 2 feet. Children often say that Germany and the United States are the same size or that South America is half the size of the United States. Upon investigation we find a full-page map of Germany and a full-page map of the United States, or a full-page map of South America and a double-page map of the United States. Because the changing scales are not taken into account, the child makes these grotesque errors.
7. Provide exercises in reading scaled drawings and maps. Let children examine floor plans issued by real estate operators. Ask as many questions as possible concerning size, distance, direction, location of windows, closets and doors, direction or exposure in which these open. Similar exercises in interpretation of maps will give children a very useful body of information.

Finding the area of a rectangle.

1. Practice in finding area of a cover of a cigar box, of the cover of a small book, and any other small surface by using a square inch and actually measuring with it. Practice in estimating similar areas without using the square inch.

2. Lead children to perceive need of finding a means of computing large areas other than by actually measuring the surface with a square inch or a square foot.
3. Teach the formula, $A = l \times w$, by using squared paper. Each side of a square is regarded as an inch, or a foot, or a yard. The area is deduced as $l \times w$, and the denomination is reasoned out. If the length and width are in inches, the area must, of course, be square inches. Many teachers prefer the following procedure in lieu of Step 3, above:
4. Children and teacher draw rectangles 2 × 3, 3 × 4, 4 × 5 inches or feet, on papers or on the blackboard. Develop a formula by analytical drawings like the following:

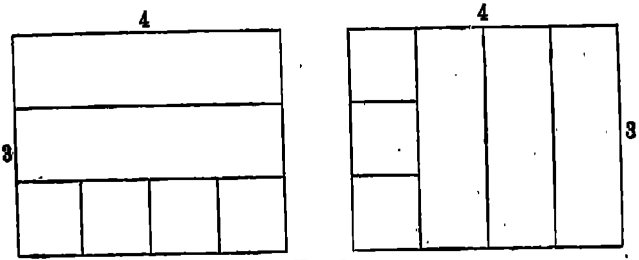
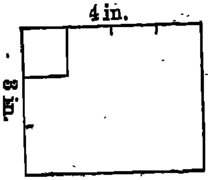


FIG. 6.

Elicit, "4 sq. in. in one row; 3 × 4 sq. in. in 3 rows, etc." After a few rectangles are so treated, lead the class to formulate—"The number of square units in one row multiplied by the number of rows," 3 × 4 sq. in. or 4 × 3 sq. in. = 12 sq. in. avoid 4 in. × 3 in. = 12 sq. in., 4 × 3 × 1 sq. in. = 12 sq. in.

Strong objection to this procedure is voiced by Thorndike. He prefers a diagram like the one to the right. "It is better actually to hide the individual square units," he argues, and builds up his position with four reasons: (1) The vital need is to get children to see that if the x and y dimensions are known, the area has xy square units. (2) The concept of "times" in these situations is obvious to most children for they rarely confuse it with addition. (3) Once the xy rule is deduced, it holds for fractional and mixed-number dimensions like $3\frac{1}{2} \times 4\frac{1}{4}$.



(4) And, finally, children must not be given the concept that a unit of area, say, the square foot, must always be square and never oblong. Surely one square foot may be produced by a rectangle 4 feet long and ¼ foot wide.[1]

5. Grade examples in area: (1) length and width given in same denominations; (2) length given in one denomination, for example, ft., width in another, for example, yards; (3) fractional forms introduced, 36½ ft. by 12¾; (4) finding the area when length and width are not given directly, for example, find the area of a floor 30 ft. long and half as wide; find area of a square floor measuring 16 ft. on a side; find the length of a floor that is 25 feet wide and has an area of 1000 sq. ft. These examples, although hypothetical, are permissible because they are interesting to children and appeal to the puzzle instinct. They give effective control of this basic principle of mensuration.
6. Introduce formula, base × altitude, as area of any rectangle.

Finding the area of a parallelogram.

1. Through a practical situation the teacher must motivate the new problem—finding the area of a parallelogram. She must elicit the importance of knowing the distinguishing characteristics of a parallelogram. Let children draw both rectangles and parallelograms having bases and altitudes of equal lengths.
2. To lead children to formulate the rule for the area of a parallelogram guide them in the following construction:

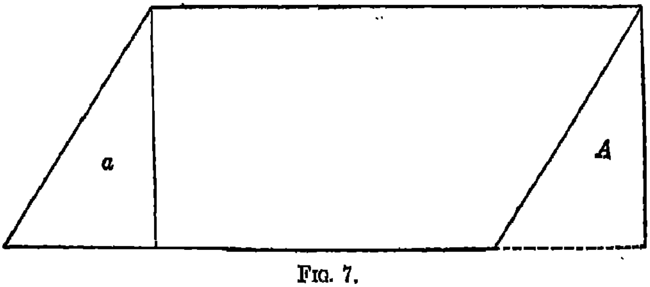


FIG. 7.

The parallelogram is drawn. The two triangles, *a* and *A*, are formed by drawing vertical lines to base. Cut out triangle *a* and superimpose it on A. Elicit: (1) the two triangles have equal areas; (2) the new figure (when a is on A) is a rectangle; (3) the area of the rectangle is the same as the area of the parallelogram. Lead children to conclude that the area of parallelogram must therefore be equal to the area of the rectangle of similar base and altitude, that is, base × altitude. *Caution:* Be sure that the children do not confuse the dotted line representing the altitude with one of the parallel sides of the parallelogram.

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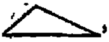
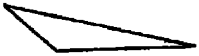
[1] E. L. Thorndike, *The Psychology of Arithmetic* (Macmillan Co., 1922), pp. 258–259.

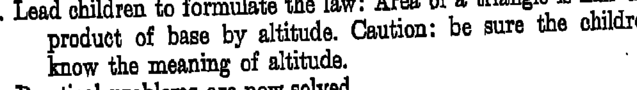
3. Have the children work practical problems.

Area of triangles.

1. Through practical problems showing the need of knowing how to compute the area of triangular surfaces the new lesson may be motivated.
2. Children draw a rectangle, a variety of parallelograms, and then one diagonal in each. The resulting triangles are now cut out and superimposed. Elicit: the area of a triangle is half the area of a parallelogram or rectangle from which it is made.
3. Children and teacher draw a right-angled triangle, a triangle having shape of , and another somewhat like



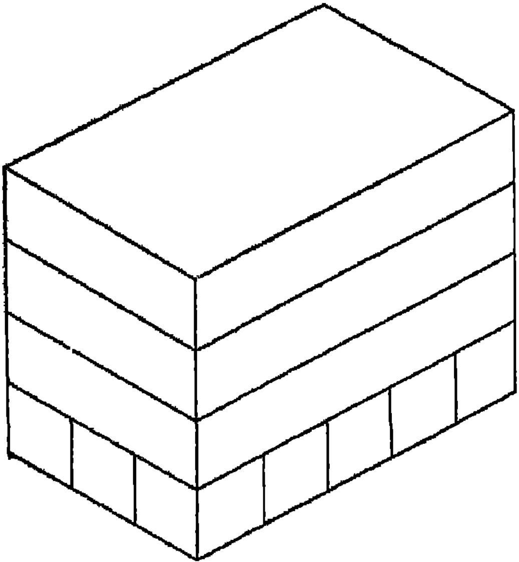
Complete the rectangle or the parallelogram as follows:



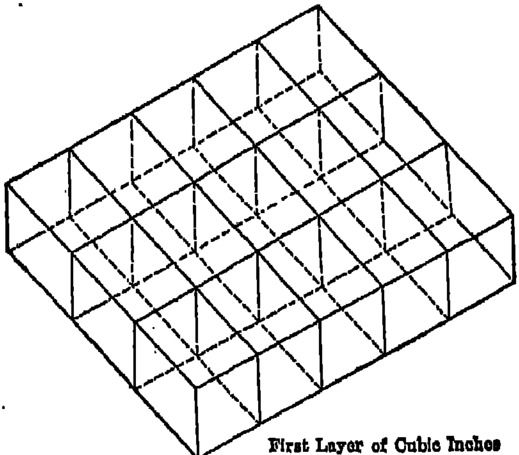
Triangles are now cut out and the same conclusion is elicited, namely, the area of a triangle is half the area of its rectangle or parallelogram.

4. Lead children to formulate the law: Area of a triangle is half the product of base by altitude. Caution: be sure the children know the meaning of altitude.
5. Practical problems are now solved.

Finding the volume of rectangular solids. The plan suggested for teaching the area of a rectangle can be followed, with suitable modifications, for teaching the volume of rectangular solids. At first an attempt is made to find the content of small boxes by using a cubic inch. This is soon given up because of the great inconvenience, and the class is led to realize that a method of ascertaining the content without the



Box 5 x 4 x 3
FIG. 8.



First Layer of Cubic Inches
FIG. 9.

tedium of using a cube unit is necessary. In developing the law we must avoid, "3 in.×5 in.×4 in.=60 cubic inches."

The law is derived by finding that in a box 5×4×3 inches there are 15 cu. in. in the first layer of cubic inches and that there are four such layers of cubic inches. To teach this law concretely blocks are necessary, which are laid in four layers of fifteen blocks each. After the law is developed, thoroughly understood, and applied to a number of examples, the formula "length×width×height" or $l \times w \times h$ is given as the short form.

The circle. Much difference of opinion exists concerning the wisdom of teaching the circle before this plane figure is met in geometry. In the last ten years the circle has gradually been eliminated from the courses of study in arithmetic. Many school people insist on teaching radius, diameter, and circumference, but they are ready to postpone to the high school all consideration of the area of the circle.

To teach children in the junior high school the meaning and relationships among radius, diameter, and circumference is no difficult task. By drawing circles, radii, and diameters, the pupils soon learn (1) the meaning of radius and diameter; (2) that all radii and all diameters of a circle are equal in length; (3) that a radius is half of a diameter. These facts are readily obtained from the class.

It is not difficult at this point to teach that the circumference is πd. A hoop is rolled on a straight line on the floor, starting at the point X on the hoop and ending when the X once more touches the floor. The distance traversed by the hoop is compared with the distance across its widest part, and the resulting ratio of $3\frac{1}{7}$ is easily found.

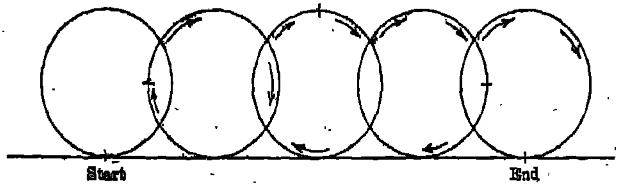


FIG. 10.

If the area of a circle must be taught, direct telling is perhaps the wisest procedure for classes below the senior high school. Assuming that the children know πd, the teacher tells them that the area of a circle is πr^2. The rest of the available time is spent on application. The rationalization offered these young children is usually beyond them and lacks the dignity of a mathematical proof. Most authorities would therefore simplify the course in arithmetic by deleting the circle from the work of the first nine years. Those who will study secondary-school mathematics will learn a mathematical proof of πr^2; the others just do not meet πr^2 in their schooling. The areas and cubical contents of figures other than those we have treated are, by common agreement, postponed to the senior high school.

PROBLEMS FOR STUDY AND DISCUSSION

1. Formulate a series of graded exercises, suitable for the early grades, designed to develop the concept of measurement. Be sure to indicate the nature of the objective material to be used, the changing character of the units, and the successive ideas to be developed.

2. Invent two practical problems which require the use of each of the fundamental operations in denominate numbers; invent two problems that require reduction ascending, and two that call for reduction descending. If you find difficulty in carrying out the directions, consult a textbook in arithmetic. What is your estimate of the problems you found in the textbook? What, do you think, is the purpose of this exercise?

3. The nature of the units of measure is often an index of the character of the industrial and economic life of a people. Plan a discussion of this idea. What references will you consult before beginning your plan? (See classified and annotated bibliography in the Appendix.)

4. Too often units of measure remain empty words because .... Complete this statement and then set forth as many suggestions as you can to guide in teaching three standardized units.

5. In teaching a table of measure, it is not advisable to present it in its complete form to the class. State your agreement or disagreement with this statement, by indicating, step by step, how you would teach all the facts in a table, say linear measure.

6. Two common errors are found in the children's work in denominate numbers. What are these errors and what measures may be employed to anticipate or to eradicate them?

7. Are there any suggestions, directions, or statements in this chapter with which you find yourself in disagreement? What are they? What references will you consult before formulating your own position?

CHAPTER XXX

TEACHING BUSINESS FORMS

Place in the course of study. Business forms merit as important a place in the course of study as they have in actual business. Incidental attention to the common business forms can no longer be justified. Properly stressed, they may be used to motivate many of the operations that must be taught.

Business forms to be taught in the elementary course. The business forms that must be accorded a place in the course of study in arithmetic are given below in the order in which they may be taken up.

1. *Bills.*
    - *a.* For articles purchased on same day.
    - *b.* For articles purchased on different days.
    - *c.* Bills receipted in full payment.
    - *d.* Bills showing payment on account.
    - *e.* Bills for labor or services rendered.
    - *f.* Bills showing a discount or a double discount.
    - *g.* Bills showing credit items allowed, that is, a person is charged $25.40 for 6 articles purchased, but allowance is made because 2 of these, costing $6.10, were returned.
2. *Invoice.* Children must be helped to understand that an invoice is a memorandum of goods shipped, showing quantity, quality or style, and mode of shipment, so that the purchaser may be able to check up what he received. A bill, on the other hand, gives most of this information, but comes in the nature of a request for payment.
3. *Monthly statement.*
4. *Orders.* In teaching children to order goods catalogs should be used, and the necessary information to be included in the order must be formulated before the order is written. Children can be led to see the need of telling (*a*) the exact date the article is ordered, (*b*) its catalog number, (*c*) quantity and quality,

such as color or size, etc., (*d*) date of shipment, (*e*) mode of shipment, (*f*) place to which goods are to be sent, and (*g*) mode of payment. It is not always necessary to give all this information, for items (*d*) and (*e*) may be of no importance in some forms of business; item (*f*) must be given when goods are sent to warehouses or to a customer of the firm making the order, and item (*g*) applies only in limited cases.

5. *Receipts*. In making out receipts vary the conditions, that is, give receipts in full payment, on account, and for goods received, as well as for money. In all cases children should make out the stubs as well as the receipts.
6. *Checks*. In connection with the study of checks the following conditions should be included: (*a*) making out stubs; (*b*) paying to the order of a firm, or to the order of John Jones, Treasurer, not John Jones personally; (*c*) paying to order of cash; (*d*) indorsements; (*e*) certified checks; (*f*) how a check serves as a receipt.
7. *Deposit slip*. Fill out slip showing bills, specie, and checks ready for deposit in a business bank.
8. *Cash account*. Page from cash book of any retail dealer.
9. *Household account*. Principle of cash account applied to household receipts and disbursements.
10. *Personal account or ledger account*. The term *personal account* may refer either to (*a*) an account kept by one person to show his private receipts and disbursements, or (*b*) to a ledger account showing the purchase and payment status of any one customer. In business the usual significance of personal account is the latter.
11. *Post-office business*. Copies of pamphlets or circulars giving the mailing rates, the classes of mail, parcel-post rates and money-order rates should be obtained by the children. After the information contained in these pamphlets has been read understandingly, a variety of examples may be solved, of which the following are illustrations. It is obvious that the tables of rates and zones should not be memorized.
    - *a*. Find the cost of sending a letter or first class mail weighing 5½ oz. from New York to Boston.
    - *b*. What would be the cost if the letter were registered? If sent by special delivery?
    - *c*. A Sunday newspaper weighing 7½ oz. is sent from your post office to San Francisco. What amount of postage is necessary?
    - *d*. A package of shoes weighing 1 lb. 4 oz. is sent from your post office to Sandusky, Ohio. What will the cost be?

   e. What will it cost me to send my brother in St. Louis $125 by money order?

12. *Notes.* Include "on demand" and "promissory note with interest."
13. *Single-entry bookkeeping.* Really nothing more than a coördination of cash account, personal (ledger) account, and previous information.
14. *Travel information.* Children must learn to read time-tables to ascertain travel costs and distances, and to acquire that body of information issued to travelers by transportation companies. An examination of road maps given so generously to automobile drivers will reveal a host of interesting data out of which practical problems can be made.

Suggestions for teaching business forms. *Use actual forms.* As far as possible real forms should be used. This does not preclude the use of bills, invoices, and notes ruled by the children. But in the main real bills, real checks, and real deposit slips should be obtained from business houses and banks for this work. All firms do not use bills identical in form; all banks do not follow the same arrangement of items on checks. To bring real business forms into class prepares the child for any arrangement that he may meet later.

Dramatize business procedures in which these forms are necessary. Instead of calling upon the children to imagine business conditions that make the use of these forms necessary, it is better to dramatize these situations. One child sends the other a bill; the latter compares it with his invoice and then writes his check. It is arranged to have the same child receive a number of checks which he indorses and enters upon a deposit slip preparatory to going to the bank. A sends B three bills and finally a monthly statement. B checks the monthly statement, discovers an error, and writes A, who acknowledges the overcharge and sends a new statement with proper credit items. The possibilities for this sort of dramatization are numerous and the opportunities for correlation with composition are manifold.

Careful gradation and correlation through the grades. No one business form should be taught exhaustively in one grade. In the fourth year the bill contains only whole numbers; in the fifth, the items of the bill contain fractions; in the seventh, discount and credit items are introduced. In this progressive development the various business forms are reviewed, each repetition presents a new point of view, and practice is given in the fundamental operations with whole numbers, fractions, decimals, and percentage.

Lead children to apply the work personally. Much can be done to impress these business forms on the children by encouraging each one to keep a personal account and to assist his parents, when possible, in making out bills, checks, receipts, and statements.

Give reasons for arrangement of these forms. When necessary, reasons should be given justifying the facts and the arrangements in these business forms. Children should know why checks and stubs are numbered, why one writes, "Twenty-three and 25/100________________Dollars," why there is a line between the amount and the word "Dollars," why the form 25/100 is used, why the amount is written again on the check in figures, why one must begin writing the amount close to the left edge of the check—in a word, the conditions from which existing business forms and arrangements seem to have been evolved.

CHAPTER XXXI

TEACHING THE ELEMENTS OF STATISTICS

With the unprecedented application of science to every activity of life, the graphic representations of numerical values are becoming very common. One finds statistical tabulations and graphs in advertisements and in newspaper articles, as well as in more serious reading material. The simplest of the statistical techniques must therefore be taught in the elementary and the junior high school.

Teaching graphs. Three well planned steps must be taken to make clear to young people the significance of the graph, the symbolic representation of the quantitative aspect of facts or data. It is well to start by posting a chart or graph of the progress of the class in regularity of attendance. As he plots the successive parts of the graph each day, the teacher explains the method of determining each new point of the graph. Other records are kept—the weekly class average in certain studies, weight gains of certain children, and the progress in the broad jump of those preparing to represent the class in the school athletic meet. In each case the record is reduced to a graph, the significance of the vertical and horizontal axes is explained, and the mode of plotting the new sections of the graph is shown.

On every suitable occasion children should be asked to interpret the graph as far as it has been carried by the teacher. When they show familiarity with the meaning of the graph and its method of construction they are encouraged to keep individual records: daily changes in temperature, health data, progress in spelling, progress of a community fund, etc. No full lesson is given to this work. It is introduced incidentally and discussed for a few minutes. The teaching procedure may therefore be summed up as follows:

Step 1. The teacher constructs and interprets the graphs.
Step 2. The teacher constructs and the pupils interpret graphs.
Step 3. The pupils construct and interpret graphs.

Preparatory to teaching bar and circular graphs, familiarize the children with picture graphs first. The following table sums up the sales of a particular automobile: 1930, 54 000; 1931, 74 000; 1932, 80 000; 1933, 150 000. In a picture graph, these facts would be represented as follows:

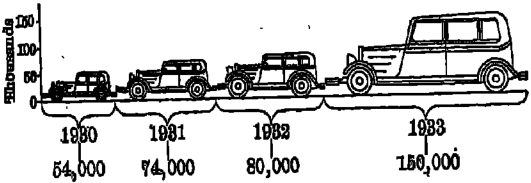


FIG. 11.

In the bar graph, each composite automobile in the preceding representation becomes a shaded rectangle. The picture graph affords an apperceptive background for an understanding of the bar graph.

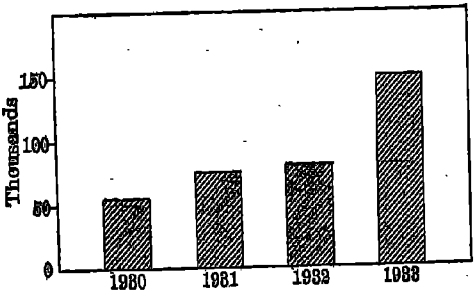


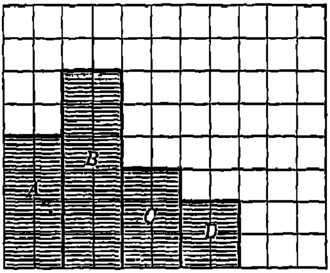
FIG. 12.

To facilitate the comprehension of the circular graph, begin with an analysis of expenses by apportioning the cost among the several items of expenditure. A community finds that out of each dollar spent, 20 cents goes for education, 10 cents for fire protection, 15 cents for police protection, 10 cents for charity and relief, 12 for recreation and parks, 10 cents for roads, 15 cents for interest on its bonds, 5 cents for prisons, and 3 cents for miscellaneous expenses. Children draw a silver dollar and proceed to apportion it among the several items of expense.

Before making a graph the data should be arranged in tabular form. As the graph progresses, the child sees that he is interpreting his numbers and their relations and helping others see them as he does. There is no dearth of suitable situations that can be utilized by children for working out graphs: individual progress in certain subjects; fortnightly gains in height and weight; athletic records of teams; interesting facts of geography and history; the daily temperature records of a month; the record of attendance and punctuality; in a word, any experience that seems to have meaning for children.

For classroom demonstrations it is well to secure a large beaver or compo board suitably framed. On it is fastened a sheet of white paper ruled like ordinary graph paper, but with larger squares. The facts to be represented vertically or horizontally are written on cardboards and fastened on the paper. Thumb tacks are then inserted at each point in the developing line graph. A string running from one thumb tack to another represents the graph. When two or three graphs are to be plotted using the same vertical and horizontal axes, two or three different colored strings and thumb tacks are used. When the calculations are no longer necessary, the tacks, strings, and descriptive matter along vertical and horizontal axes are removed and the same background of graphed paper may be used again and again. In addition to economy of time and effort, this device enables the teacher to present the work on a scale large enough to be seen by every pupil.

The uses of graphs suitable for Grades 5-9. 1. *In making diagrammatic representations.* In a group of 60 children, 16 received *A*, 24 *B*, 12 *C*, and 8 *D* in proficiency. Represent the distribution of the proficiency of the group diagrammatically.



Each box represents 2 children
A, 8 boxes, represents 16 children
B, 12 " , " 24 "
C, 6 " , " 12 "
D, 4 " , " 8 "

FIG. 13.

Upon examining a herd of cows it was found that rating the perfect cow 100 points, the cows were marked as follows: 10 at 90 points, 5 at 80, 15 at 75, 20 at 70, 20 at 65, 10 at 60, 15 at 55 and 15 at 50. Represent diagrammatically the distribution of the herd.

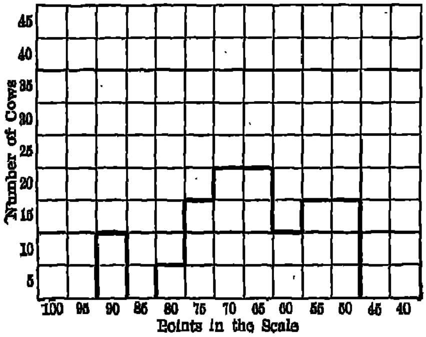


FIG. 14.

A school opened February, 1930, with 1702 pupils. Its attendance for the successive terms was as follows: 1721, 1770, 1810, 1840, 1865, 1870, 1865, 1868, respectively. Chart the increase or decrease of the school, term by term (Fig. 15).

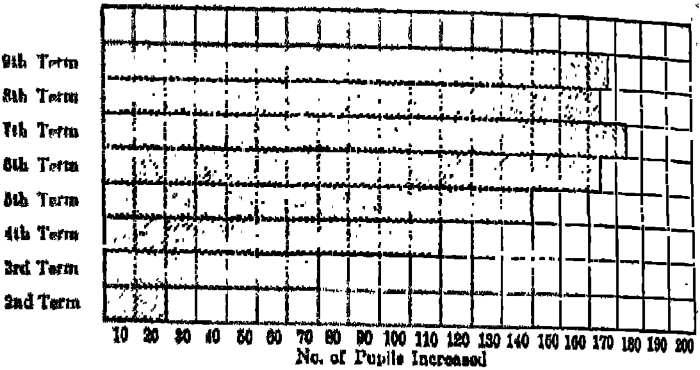


FIG. 15.

2. *Making graphs of single or double records.* A trolley car makes its round trip on an average of once very hour. The number of passengers it carried each round trip, beginning at 1 A.M. was 40, 36, 30, 26, 21, 42, 80, 116, 92, 85, 81, 80, 124, 122, 104, 92, 80, 61, 52, 57, 68, 60, 61, 55, respectively.

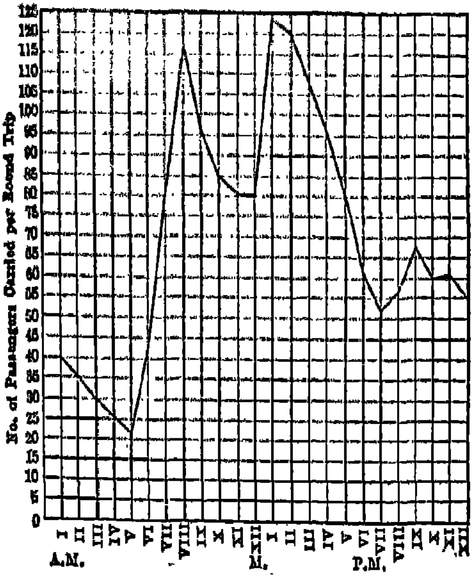


FIG. 16.

Make a graph showing the changes in the number of passengers, hour by hour. (See Fig. 16.)

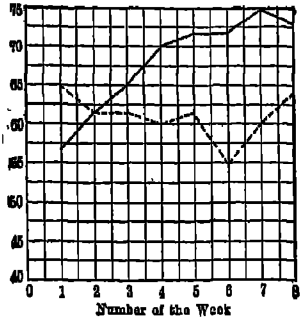


FIG. 17.

Heavy line represents first child. Dotted line represents second child.

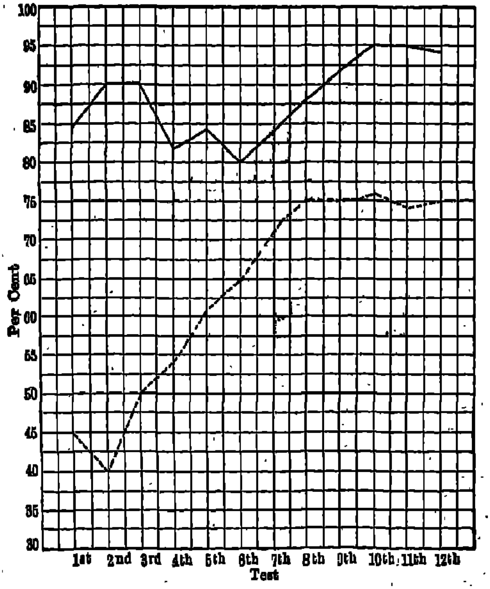


FIG. 18.

Graphs showing progress in arithmetic (heavy line) and grammar (dotted line) of a boy who liked arithmetic but thought he could never learn grammar.

A pupil who failed in arithmetic the first month of the term made the following progress for the next 8 weeks: 58%, 62%, 65%, 70%, 72%, 72%, 75%, 73%. His classmate, who made a passing mark the first month, received the following ratings in the same 8 weeks: 65%, 62%, 62%, 60%, 62%, 55%, 60%, 64%. Show, by graphs, the quality of the work of each of the two pupils (Fig. 17).

3. *Interpreting graphs.* Figure 18, page 435 is presented to a class in order to give practice in interpreting the lines, reading the numbers, and in drawing tentative conclusions. Nothing in addition to the information contained below should be given to the children.

After studying the two graphs children should state:

a. The ratings in each test in each subject.
b. The pupil's discouragement about language work at the beginning of the term.
c. The effort that this boy evidently made.
d. The progress in language work.
e. The improvement in arithmetic as well as in language work.

Teachers will find children intensely interested in such exercises, especially when they are given in friendly rivalry to find who can "see" most in the graph.

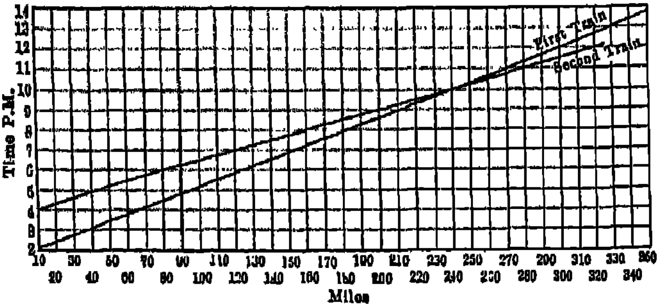


FIG. 19.

Second train overtakes first at 10 P. M. at a place 240 miles from New York, the starting station.

4. *Solving problems.* By the aid of graphs children can readily solve problems of the following type.

A train leaves New York at 2 P.M. and travels 30 miles per hour. At 4 P.M. another train leaves the same station at the rate of 40 miles per hour. At what time will the second train overtake the first?

The mode of solution of this example is illustrated in the graphs in Fig. 19.

Statistical facts and techniques for Grades 5–9. Three simple statistical techniques may be taught in the elementary and junior high schools—*tallying, arranging a distribution,* and computing one measure of central tendency, the average or the mean. No set lesson need be planned on tallying. When children vote for class officers by secret ballot, they may be shown a convenient mode of counting votes, thus, ~~1111~~, ~~1111~~, 1111=14. In a subsequent discussion various situations calling for tallying may be listed: taking inventory of the stock in a store, checking the number of each article sold by a merchant, checking a set of test papers to ascertain the number of pupils who worked each example correctly, and the like. The superiority of the system of tallying by 5's, ~~1111~~, over the individual listings without grouping, 111111111, is obvious even to inexperienced school pupils.

Arranging quantities in a convenient distribution should be taught in the same way when an actual classroom situation requires such a tabulation. It may be necessary to determine how well the class did in a particular test. The teacher reads the grades, one after the other, and the children are led to realize that no inference can be drawn from the list. A few well chosen questions will help the children to see that they must know the number who had perfect scores, then those who had 95%, then 90%, etc. The teacher suggests that these scores in intervals of five be listed, beginning with the highest, and that a tally be made of each score, as shown below.

Grade	*Number Who Made It*	
100	111	3
95	1111	4
90	111	3
85	~~1111~~ 11	7
80	~~1111~~ ~~1111~~ 1	11
	etc.	

Gains in weight for a period of two months, the rainfall of each week for a given season, and similar data may be arranged in similar ways. Children realize the value of making such systematic arrangements of quantities before attempting to use the facts.

In teaching computation of averages, two types must be differentiated. The first is illustrated by such examples as "A's weekly earnings are 12, 14, 11, 12, and 9 dollars, respectively, for each of 5 successive weeks. What is his average weekly wage?" Here the salaries are added, giving $58 for 5 weeks; the average earning per week, therefore, is $58÷5, or $\$11\frac{3}{5}$, or $11.60. Children may be required to average their marks, knowing their ratings in each of six subjects; to average *the daily class attendance for* the week, knowing the attendance each day. In this form of averaging there is little to explain and nothing that really challenges the class.

The second type involves reduction of different elements to a similar basis. Thus, all 6B children in a school were tested; their results are listed below:

4 pupils had 50%
3 pupils had 55%
8 pupils had 60%
2 pupils had 65%
15 pupils had 70%
14 pupils had 75%
5 pupils had 80%
3 pupils had 85%
4 pupils had 90%

What was the average grade?

Statistical facts and techniques beyond the three we have illustrated are clearly not indicated for pupils through the ninth grade. Experience proves, however, that these young children do understand what we have outlined and that they recognize its importance.

CHAPTER XXXII

TEACHING PROBLEM-SOLVING

Problem-solving vitalizes arithmetic by infusing purpose into its computations. Problems intensify both the utilitarian and disciplinary values of arithmetic by providing quantitative interpretations of social, economic and civic experiences. In problem-solving the child finds the real challenge of arithmetic.

Meaning of the term problem. The term *problem* must be carefully differentiated from *exercise* and *example*. An exercise is a direction to perform an operation. "Find the products of 284⅓×75, 284⅓×705, 284⅓×.75," is a legitimate assignment in arithmetic, but it is an exercise and nothing more.

An example is an exercise clothed in words so that the task seems more social and becomes, therefore, more interesting. "Find ⅔ of 24," is an exercise, an indicated arithmetical procedure. We turn it into an example by asking, "A class in which 24 pupils had made interesting picture books sent ⅔ of them to a children's hospital. How many picture books did the sick children receive?" No great penetration is required to decide that the answer is obtained by finding ⅔ of 24. But the example does not tell the child directly to find ⅔ of 24.

A problem is a challenging situation that invites solution. It cannot be resolved without careful analysis and planning. Most children would find the following situation a real challenge to their abilities:

> Which radio would you advise Mr. Smith to buy? Radio A sells for $79.50 with tubes and free installation. Radio B sells for $65.75 with additional charge of $8.75 for tubes and $5.50 for installation. As far as Mr. Smith can find out, the radios are equally good instruments.

These distinctions among exercise, example, and problem cannot always be maintained. What is a problem to one person may be only an example to another. Problem and example must be judged relatively.

Factors determining ability to solve problems. Six major factors determine ability to solve problems in arithmetic. The first is intelligence—native ability to recognize quantitative relationships. There is a closer relationship between reasoning in arithmetic and general intelligence than between reasoning in arithmetic and skill in number manipulations. While children with low I. Q.'s, 80 or less, will probably never learn to solve arithmetical problems, we must not make too fatalistic an inference and conclude that in dull normal children ability to reason in arithmetic cannot be improved. Systematic and graded instruction, with adequate practice in ways of analyzing arithmetical situations, does produce greater skill in dealing with type forms of problems.

Among the other factors we must include (1) skill in silent reading; (2) familiarity with the technical language of arithmetic; (3) understanding of situations that give rise to arithmetical problems; (4) a high degree of skill in the fundamental operations; and (5) an attitude towards accuracy that leads to questioning and checking of answers. These six factors determine, in the main, the extent to which a pupil will learn to apply effectively definite methods of attacking mathematical problems.

It is important for teachers to realize that not all children can obtain proficiency in problem-solving. Overinsistence that a child do what, clearly, is impossible for him, results in a paralyzing sense of inferiority and interferes with the development of mere manipulative skill.

Prevailing causes of error in problem-solving. Osburn[1] analyzed the work of 6000 children and found 30 000 errors in the *Buckingham Problem Test,* Form 1, as indicated in the following table.

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[1] W. J. Osburn, *Diagnostic and Remedial Treatment of Errors in Arithmetic Reasoning* (Madison, Wisconsin, State Department of Public Instruction, December, 1922).

Compare these causes with those offered by P. R. Stevenson in "Difficulties in Problem Solving," *Journal of Educational Research*, Vol. 11, February, 1925, pp. 93–103. Stevenson gives (1) physical defects; (2)

Cause of Error	*Frequency in Per Cent*	
1. Failure to comprehend the problem....	30	
2. Procedure partly correct, one or two essential elements omitted............	20	
3. Ignorance of fundamental quantitative relationships	10	Sub-total, 60%
4. Inaccuracies in fundamental operations..	20	
5. Miscellaneous	2	
6. Undiscoverable	18	
	100	Total

Osburn concludes that about 60 per cent of the incorrect answers may be traced to initial difficulties arising from unsuccessful attempts to comprehend "what is to be done." Failure in problem-solving is attributable, we saw, to (1) limited intelligence, (2) limited ability to read, (3) limited knowledge of English, (4) limited understanding of the technical vocabulary of arithmetic, (5) limited experience with situations about which arithmetic problems are made—disabilities both permanent and temporary. Time will remedy some and make others more acute. Causes 2 and 3, partly incorrect procedure and ignorance of quantitative relationships implied in the fundamental operations, may be due to faulty teaching, which neglected to develop methods of solving problems, or to inherent deficiencies in children. Cause 4, errors in fundamentals, yields to treatment by systematic, differentiated practice based on diagnostic testing that reveals individual weaknesses.

Our analysis of causes of errors in problem-solving brings us to the conclusion we reached earlier in the discussion: that not all children have the ability that insures successful solution of problems in arithmetic; that while teachers must help,

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lack of mentality; (3) lack of skill in fundamentals; (4) inability to read; (5) lack of general and technical vocabulary; (6) lack of proper methods of attacking a problem.

encourage, and exercise judicious insistence with slow pupils, they must not harass them.

Characteristics of effective arithmetical problems. 1. *Problems must be real.* The first essential of an arithmetical problem is that it set before the child a real situation. School problems in arithmetic are too often untrue to life and, to that extent, mislead the child. A textbook asks, under the caption Practical Problems: "A earned $28.84 this week, which is 16⅔% more than he earned last week. What did A earn last week?" Another book offers the following as one of a list of "practical" problems: "A dying man decided to leave ¼ of his 600 acres to his oldest son; ⅔ of what he left the oldest, to his second son; ¾ of what he left his second son, to his youngest, and the remainder to be divided equally between his wife and his daughter. What did the daughter receive?" Even those unfamiliar with the making of wills feel some doubt that they include exercises in fractions. We can easily increase the illustrations by referring to current classroom practice.

In contrast to these artificial problems let us consider the following:

> Mr. and Mrs. Morton like their house and grounds very much. They are now considering whether to continue to pay rent at $1500 a year with all service furnished by the landlord or to buy the house for $18 000 of which they will have to pay $8000 in cash and take a mortgage of $10 000 with interest at 5%. The house will necessitate the following additional annual expenses: Taxes, $320; insurance, $32.00; repairs, about $35.00; heating and hot water, $450. What would you advise the Mortons to do, rent or buy?

In a socialized recitation the total annual cost of living in the house under the purchase plan is ascertained:

Loss of interest on the $8000 @ savings bank rate of 3% is	$ 240.00
Interest on $10 000 mortgage @ 5% is	500.00
The other annual expenses total	837.00
Total annual outlay	$1577.00

The teacher now leads the class to consider why the comparison of $1577, the estimated annual cost, with $1500, the annual rental, will not give the Mortons an adequate basis for coming to a decision. Other factors involved in the decision are: the responsibility for unexpected repairs, the expense of repairs that do not come annually like interior and exterior painting and roof replacements, the freezing of $8000 of cash and the obligation to carry a $10 000 mortgage, and the possibility of appreciation or depreciation of values. Clearly the arithmetical totals in this problem are less important than the other considerations. Such a problem, thoroughly discussed with an eighth grade, leaves the children richer in an understanding of business and social relations.

> Can I afford to own an automobile? Each year I earn $3200 and save $600. I have $2750 in the savings bank. The cost of the automobile and its care would be as follows: the car, $960; insurance, $120 a year; registration fee and operator's license, $16.50 a year; garage $7 a month; cleaning and repairs, about $75 a year. At the end of three years I can sell the car for about $350.

Let us assume that the class has made the following calculations:

Cost of the car	$960.00
Insurance for 3 years	360.00
Registration and license for 3 years	49.50
Garage at $84 a year, for 3 years	252.00
Cleaning and repairs for 3 years	225.00
Total cost for 3 years	$1846.50
Less receipt from sale of car	350.00
Net cost for three years	$1496.50
Total savings for 3 years	1800.00

The answer is ________

Interesting discussion is assured: Is it wise to sink almost all savings, about $1500 out of $1800 in a car? How old is this person? Is the answer the same regardless of age? How about the cost of gasoline? This person probably travels especially during vacations and free time. Will the automobile save him travel costs or increase them? After careful consideration of these questions, the children realize that the arithmetic result is not the sole consideration and that a decision can be reached only when a situation is thought through.

A grocer who pays 12 cents for a can of fruit charges his customers 18 cents. One of them objects that a profit of 6 cents per can is too much. The grocer explains: "You forget that to the cost of each can I must add other expenses." What other expenses did the grocer name to justify his price of 18 cents per can?

Few children know that the small shopkeeper has a large overhead cost and that each article sold must carry its share of that expense. To let children believe that the difference between the cost and the selling price of an article is clear profit is to give them a serious misunderstanding of common business practice.

Other illustrations of true or real problems follow:

John's average in arithmetic for October was 78¾, for November, 82¼; William's average for October was 81.75, for November, 83.25. Who made greater progress? By how much?

An automobile is advertised to run on an average of 15 miles on a gallon of gasoline. Mr. Jones, eager to find out whether his car is doing what the advertisement promised, kept a record of the daily runs of his car and the amount of gasoline used up:

Day	Amount of Gasoline in Tank at Start	Amount of Gasoline in Tank at End of Day	Miles Run That Day	Amount of Gasoline Used That Day
Monday....	14 gallons	8½ gallons	66.5	..... gallons
Tuesday....	8½ "	2¼ "	75.4	..... "
Wednesday.	12¼ "	5¾ "	78.6	..... "
Thursday...	5¾ "	1¼ "	54.2	..... "
Friday.....	11¼ "	2½ "	105.6	..... "
Saturday...	10 "	3¼ "	81.8	..... "
		Totals.....	?	? gallons

a. Fill out the last column for Mr. Jones.
b. How many miles per gallon did his car run each day?
c. How many miles did his car run that week?

d. How many gallons of gasoline did his car use up that week?
e. How many miles per gallon did his car average that week?
f. What do you think Mr. Jones said to the manager of the automobile sales company? What facts did he show to prove his case?

Walter liked to read the mileage record on his father's car. At the beginning of the trip he read 5249.8; at the end, 5624.5. How many miles did the family ride on that trip?

Susan's mother was making jelly and needed 32 cups of sugar. Susan measured the sugar in the sugar tin for her mother and found that they had only 28¼ cupfuls. Her mother asked her to borrow what was needed from their neighbor. How many cupfuls did she ask for?

The test of a good problem is quite obvious: does it present a *situation that is likely to arise in the life of a child?* What is a good problem for children in an agricultural community is not necessarily suitable in an urban or in a mining center. The needs of the life that the child sees about him determine the appropriateness of the problem we ask him to solve.

An interesting question arises at this point in our discussion: must a good problem have its origin solely in the child's actual experiences or may it transcend the limited scope of the child's life? To formulate arithmetical problems wholly around the immediate life of our pupils places undue limits on the teacher and tends to make the problems commonplace. Surely we must seek situations other than those concerned with tag, marbles, tops, and pies. The vital question always is whether the background of the problems comes from the life which the pupil lives or must learn to live in his community. Problems of the counting house and the factory not in the present experience of the child are therefore appropriate and constitute instrumentalities for breaking the confining bonds of the child's own circumscribed life.

An occasional hypothetical problem must not be condemned. An imaginary situation may appear more attractive and more vital to a child than one that is genuine. Fiction may seem more real than history, and fancy may be more vibrant with life than fact. The hypothetical problem may take on the aspect of a puzzle and thus become a very effective challenge to the child. That explains why the old hare and hound problems seemed real enough to us in our youth and why we struggle willingly to-day with the problem of the man who rowed against the stream. Used in moderation, the hypothetical problem is defensible.

2. *Conditions in problems to be true.* In formulating problems it is essential that costs, distances, speeds, capacities, and the like should approximate the facts of life. Children should not be asked to find the cost of 254 bales of cotton at $1.60 a bale; to find how many miles per hour a train runs that covers a distance of 189 miles in 2¼ hours; to find how many square feet of wall and ceiling will be plastered in a room 20×15×9½ feet; to find the premium on a life-insurance policy of $3550 at ⅜% premium; to find how far a man walks in 4½ days who walks 4 miles in one hour. All of these problems, taken from school assignments, embody assumptions and conditions that are false; cotton at $1.60 a bale is ludicrous; a regular train that runs 84 miles an hour or 189 miles in 2¼ hours is still a possibility of the future; a room must have at least one door and one window; life insurance policies do not set so fixed a sum as $3550, nor is the premium computed at a fixed percentage of the face of the policy.

A typical illustration of the divorce between classroom and business practice is found in the problems that are usually taught under the head of *partnership* in commercial arithmetic. The problem reads as follows: A and B enter into partnership, A investing $2000 and B $3000. At the end of the year it is found that the earnings of the business is $4500. What will be the share of each? The usual solution takes a ratio form:

A deposits	$2000
B deposits	3000
A and B deposit...	$5000

A receives 2000/5000 or 2/5 of the earnings or 2/5 of $4500 which is $1800
B receives 3000/5000 or 3/5 of the earnings or 3/5 of $4500 which is $2700

B receives $900 more than A. But it must be assumed, since nothing is said to the contrary, that A and B invest equal time and skill. According to the problem, the only difference in the investments of the two partners is B's additional one thousand dollars. For investing $1000 B receives $900 or 90 per cent interest. The situation is not only unbusinesslike but also dishonest. Actual business practice would give to A 6 per cent on his $2000 and to B, 6 per cent on his $3000, or $120 to the former and $180 to the latter. This money, $120+$180 or $300, would be deducted from $4500 and the remainder, $4200, would be divided equally between A and B. For investing an added $1000, B receives an added interest of $60, a fair return. Hence B's total share would be $2100+$180, or $2280; and A's, $2100+$120, or $2220.

An effective arithmetical problem gives knowledge which supplements daily experience by emphasizing its quantitative aspect. The problems cited above add to the child's stock of misinformation.

3. *Language of problems.* A problem must obviously be phrased in language that is clear, simple, and attractive. To word a problem correctly requires, among other things, a knowledge of the extent of the pupil's vocabulary, a ready control of a sufficient number of appropriate business or practical settings that are true in every detail, and finally fluency of expression. Formulating problems is a surprisingly exacting exercise in composition. Teachers of experience appreciate the worth of a good textbook that is replete with vital problems.

At times the very wording makes or destroys a problem. "Every problem is less of a problem to the extent to which it can be solved by memory of the method by which similar problems have previously been solved. The expression 'drill in problem-solving' . . . is therefore a misnomer, for, in so far as previous experience—in the form of the drill referred to—enables an exercise to be solved, that exercise is not a problem."[2] It becomes apparent, therefore, that problems must be so worded that each succeeding one does not follow the pattern of its predecessor. We sometimes find a series of problems like the following:

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[2] E. R. Hamilton, "Insight and Skill in Arithmetic," *Journal of Educational Research*, Vol. 12, September, 1925, p. 139.

> A man had $15.75 and lost $5.25; what per cent of his money did he lose?
>
> A boy who had 84 marbles lost 21; what per cent of his marbles did he lose?
>
> An auto tire guaranteed to stand the wear of approximately 20 000 miles lasted only for 8000 miles; what per cent of its expected use did the purchaser enjoy?

In all of these problems the first number is the denominator and the second the numerator.

Problems must be so varied in sentence structure that the pupils are forced to think before deciding on the solution of each successive problem. Let us consider an illustration:

The Type:

A car averages 15½ miles on a gallon of gasoline; how far will it run on 124 gallons?

The Variations in Language Only:

1. How far will 124 gallons of gasoline take me in a car that averages 15½ miles to a gallon?
2. If a car can run 15½ miles on one gallon of gasoline, how far will it run on 124 gallons?
3. How long a trip may I plan with 124 gallons of gasoline in a car that averages 15.5 miles to a gallon?
4. I used up 124 gallons of gasoline in a car that usually makes 15.5 miles on one gallon. How long a trip did I take?
5. When will a supply of 124 gallons of gasoline be used up if I can get 15½ miles out of 1 gallon in my car?

Monroe made as many as 28 variations of the problem, "Goods cost $1.50 a yard; the amount spent for material is $7.50; how many yards were bought?[8] Of course, no teacher need seek that degree of variation, but it is important that teachers and textbook writers give earnest thought to this phase of the presentation of problems lest pupils become dependent on a stereotyped form of expression.

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[8] W. S. Monroe, "The Derivation of Reasoning Tests in Arithmetic," *School and Society*, Vol. 8, September 7 and 14, 1918, pp. 295–299, 324–329.

How briefly or how elaborately should the problem situation be presented? The question is important, and the answer, although not absolutely definite, inclines decidedly towards fullness of statement. An investigator gave these two forms of the same problem to 513 children in the fifth grade:

Form 1. After traveling 160 miles a man has 4 gallons of gas left in his automobile. How many miles did he get to a gallon of gas if he bought 8 gallons on the way and had 6 gallons when he started?

Form 2. Last summer Agnes Purdy, her brother, Archie, and their parents took a trip in their Ford. Archie measured the gasoline when they started. "We have 8 gallons," he told his father. At the end of the day he found 4 gallons of gasoline in the tank. They had bought 6 gallons at a station on the way, and had traveled 160 miles. Agnes told her mother that they had made ______ miles to a gallon that day.

The results are interesting: 197 pupils (38 per cent) solved the first one correctly, and 253 (49 per cent) solved the second correctly. Myers ascribes this difference in results to the fuller and more interesting statement in Form 2.[4] He may be correct in his interpretation, but there are other explanations of the superior performance by the children. The second form is more direct than the first; it is surely more chronological. Undoubtedly the greater attractiveness of the narration is a factor in focusing attention of the children.

Sources of material for problems. To list the sources of materials for problems resolves itself into an enumeration of almost all the vital activities of life. A few typical sources are: home economy, personal earnings, occupational experiences, recreational activities, geography, history, elementary science, drawing to scales, statistics and graphs, school and class activities, hypothetical situations, social and civic activities—the gamut of life. The wider the range of sources from which problems are taken, the surer are they to approximate the needs of everyday life.

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[4] G. Cleveland Myers, "Imagination in Arithmetic," *Journal of Education*, Vol. 55, June 13, 1927, pp. 662–663.

Monroe made a fourfold classification of problems in arithmetic and then analyzed four commonly used textbooks to ascertain their distribution of problems. His findings are tabulated below:[5]

Source of Problem as Formulated by Monroe	Percentage of Problems Based on Each Source	
1. Relating to occupations		
a. Agriculture, forestry and animal husbandry	10.8	
b. Manufacturing and mechanical industrial activities	18.1	
c. Transportation	9.9	
d. Trades	21.9	
e. Miscellaneous	9.3	70
2. Relating to home activities		
3. Relating to personal activities		
4. Relating to activities of school children		30

An analysis of the census figures available at the time of the study reveals the fact that 9.5 per cent of the people are engaged in trades and 33.2 per cent in agriculture. The inference that is implied cannot be drawn with impunity. What if 33.2 per cent of the people are in agriculture and only 10.8 per cent of the textbook problems are based on rural occupations? Shall we condemn the textbooks on this account? Are not many problems so formulated that their sources are incidental while their challenge is resident in the relationship among the quantities? Must economic activities and the material in arithmetic necessarily be in perfect correlation?

Wise teachers keep an eye open for suitable and interesting problems, whenever any experience suggests them. But it is hardly reasonable to expect teachers to be the authors of all or even most of the problems which their classes are asked to solve. The textbooks in arithmetic must provide a wide range of problems so that teachers may select what is most appropriate for their pupils. It is as unreasonable to expect a teacher to invent all the problems in arithmetic as it would be to demand that she write the material for the textbooks in history.

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[5] W. S. Monroe, "A Preliminary Report of an Investigation of the Economy of Time in Arithmetic," Second Report of the Committee on Minimum Essentials in Elementary Schools, *Sixteenth Yearbook* of the National Society for the Study of Education, Part I (Public School Publishing Co., 1917), pp. 111-127.

Gradation of problems. Various attempts have been made to grade problems but none seems satisfactory. The number of operations is often made the basis of gradation. The unproved assumption, of course, is that a single-operation problem is simpler than a two-operation problem. But children have consistently found the problems under A simpler than those under B.

A	B
John saved 25 cents. He then spent 10 cents for a toy aeroplane and 5 cents for a balloon. How much did he have left?	John's mother, not knowing the price of Clenso Soap, gave him 25 cents and told him to buy as many cakes as he could. At the grocery store John saw the following sign: CLENSO SOAP—SALE—4¢ A CAKE How many cakes of soap did he bring home?
Judith needed 85 cents for the birthday gift she was hoping to buy for her mother in November. She saved 45¢ in September and 35 cents in October. She decided to ask father for the rest. How much did she hope to get from her father?	Judith has 85 cents to buy ribbon for class bows. How many yards of ribbon will she buy if the kind that is needed costs 12 cents a yard?

Experience refutes the simple assumption that one arithmetical process is necessarily simpler than two. Suzzallo[6] evolved a more elaborate system of gradation, which is based on the character of the arithmetical operations as well as on their number. The following progressive steps are recommended:

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[6] H. B. Howell, *A Fundamental Study in the Pedagogy of Arithmetic* (Macmillan Co., 1914), p. 110.

The Sequence	The Operation	The Sequence	The Operation	The Sequence	The Operation	The Sequence	The Operation
1	+	6	−+	11	×÷	16	×−
2	−	7	×	12	÷×	17	+÷
3	++	8	÷	13	+×	18	÷+
4	−−	9	××	14	×+	19	−÷
5	+−	10	÷÷	15	−×	20	÷−

Our reluctance to accept this gradation is based on two considerations: (1) the plan was not evolved experimentally; (2) the language and the situation in a problem are vital factors in determining the relative difficulty of a problem. These have not been given consideration in this checkerboard arrangement of arithmetical operations.

Types of problems. To secure adequate variety of problems it is essential that children be taught at least the following types:

1. *The practice problem.* Skill in solving a new type of problem is developed by practice problems which are almost examples, because they bear a striking similarity to one another and contain a cue word that suggests, if it does not actually tell, the correct operation. The following are practice problems:

Group 1

A man who earns $4.25 a day receives how much for 6 days?

A boy who saves an average of $1.55 a month out of his allowance, has how much in his bank in a year?

A can of automobile oil weighs 10.5 lb. How much weight does a shelf bear if there are 42 cans on it?

Group 2

At the rate of 70 cents a dozen, what will 72 coat buttons cost?

Bananas are selling at 18 cents a dozen. What must I pay for 48 bananas?

What will 36 eggs cost at 21 cents a dozen?

Group 3

I buy pencils at 1 cent each and sell them for 18 cents a dozen. What is my profit per dozen?

A boy bought newspapers at 2 cents each and sold 30 at 3 cents each. What did he gain?

Apples that cost me 2 cents apiece are sold at 3 for 10 cents. What is my profit after I sell a dollar's worth?

Each group of problems is introductory and is designed to afford drill on a new mode of computation or solution. The line that distinguishes a practice problem from an example is not always easily discerned.

2. *Problems grouped around one activity.* A plan frequently employed in formulating problems is to group a series of them around one activity. The first group of problems may pertain to the clothing industry, the next to athletics, a third to transportation of freight, a fourth to cutting lumber, and a fifth to opening a small retail store or a refreshment stand. Frequently problems so grouped teach as much about an industry as about arithmetic. An illustration of one of these narrative problems, or rather of a series of problems grouped around one activity, is given below:

1. Mr. Walkers saved $4500 and decides to open a grocery store. He can have one of two equally good stores; the first at a monthly rental of $125; the second at a rental of $141 for a month for 11 months and no rental for the twelfth month of each year. Which offer should he accept?
2. He plans to put aside $2500 after fixing up the store. His expenses he finds will be as follows: Rent for the first month, $125; fixtures, $245; telephone installation, $12.25; cash register, $140; lighting fixtures, $32.60; icebox, $320; other equipment like scales, $62.50; first payment on goods, $1200. Will he be able to put aside $2500?
3. At the end of the first month he took in the following:

   For vegetables, $240.60; profit, 20 per cent
   For groceries, $560.00; profit, 16 per cent
   For eggs, milk, cream, butter, $280; profit, 10 per cent
   For other things, $260.60; profit, 20 per cent

   His total running expenses, not counting the cost of what he sells, is $210 a month. Would you say he is doing well in his business? Which one of the four classes of things he sells brings in the greatest profit in actual dollars and cents?

The situation can be made to yield problems in denominate numbers, in banking, in taxes, in any branch of arithmetic, for Mr. Walkers' grocery store can be made to prosper at will.

8. *Problems without numbers and incomplete problems.* In teaching the solution of problems we are concerned essentially with clear visualization and accurate comprehension of the conditions of the problem and with the development of sequential thinking. At such times the fundamental operations are secondary considerations. It is advisable, therefore, to give problems that contain no numbers, and to require children to tell what they would do to obtain the answer. The following are examples of this type of problem:

1. What must you know to compute the standing of a team in any competition? How would you find the percentage for a team if you knew these facts?
2. Goods can be sold at a low price and a small profit is made; I can sell half the goods with double the profit on each article. How would you find which was more profitable?
3. Goods can be bought at a discount of $52.50 cash if I pay at once, and no discount if I pay in 90 days. How would you find whether it would be better to pay cash or keep my money in the bank for the 90 days? What other fact or facts would you need to know?
4. If a room is twice as long as it is wide and you knew its length, how would you find the area of the floor?
5. A barrel contains 15 gallons, 3 quarts, and 1 pint; what else must you know to find the number of bottles that can be filled from it? If you knew this additional fact, how would you find the number of bottles that can be filled?

The class is directed to read a designated problem carefully. A few questions are asked to make clear the situation of the problem and to bring out the meaning of unfamiliar words. The solution of the problem is developed as follows:

What is to be found? What is given in the problem? What additional facts are needed? What will the first step be? How will you perform it? What will the second step be? How will you perform it? Is the answer reasonable? Why? Can the answer be proved? How?

We have here a plan to insure comprehension of the conditions of a problem and to quicken its solution. As an occasional exercise, given when a mode of thinking is to be habituated, it is of inestimable worth; as a regular practice this type of problem and its visualized solution cannot be recommended.

4. *Problems invented by the pupils.* Little value attaches to the undirected exercise in which children are asked to invent problems to be solved by their classmates. Too many puzzles may be introduced or the lesson may be filled with such a variety of problems that nothing is stressed adequately. The problems presented by the bright children are too difficult for the class as a whole, and those formulated by their dull classmates usually do not justify the time required for solution. The field or the nature of the problem must be carefully defined and delimited for the class. Such requests as the following may evoke interesting responses:

1. Invent a problem giving the cost of an article after a discount has been allowed and require the finding of the original or list price.
2. A piano listed for $1250 was bought by a school for $1175. Evidently a discount is given to educational institutions. Make a problem using these facts.
3. Mr. A must make up catalog prices. He must allow a trade discount of 15% to dealers and yet he must sell article No. 501 for $5.95 a dozen and article No. 502 for $10.20 a dozen. Use these facts and make up as many problems as you can.
4. Mr. B. bought goods from A. H. Whiting and expected his bill to be $470.40 because he paid within 10 days and thought he was entitled to a discount of 2%. What questions can you ask your classmates?
5. Train No. 42 left Albany at 9 A.M. and arrived in New York at 12.30 P.M. The distance between the cities is 149 miles. Make up two problems using these facts.
6. Mr. C. has $3500. He can put it into a savings bank which pays interest at 3%, semi-annually. His brother asks him to invest the money in his business at 6% interest annually. Make use of the facts in as many problems as possible.

Such exercises in inventing problems make a genuine contribution to the child's growth in arithmetic. In a regular problem a whole situation is given him and a definite question is asked. He knows what is wanted and what to accept as established fact. In these exercises the conditions are reversed. He may have all the facts or he may have to decide what additional facts are needed to complete a situation. He must see the usual elements in new relations.

The child who invents a problem is doing what every merchant must do. In real business arithmetical problems do not spring full grown; they are formulated as a definite need for accurate information arises. The manufacturer may need to know his rate of profit. He decides what facts to gather: cost of labor, raw material, cost of merchandising, and overhead charges. Before he solves his problem he must formulate it completely and accurately. Children must therefore be given training in synthesizing business facts into a business problem, as well as in analyzing complete situations into their constituent factors.

5. *Problems that make up a whole project.* In an earlier discussion of project-teaching, illustrations were given of problems that constitute the backbone of a project. Projects may deal with the following experiences: the organization of a school bank; the establishment of a school coöperative store; the opening of a new neighborhood store; the cost and the assessments for a civic enterprise; the distribution of population in the United States; the extent of railroad trackage; the use of the automobile or the telephone in the United States as compared with other countries. These are further illustrations of projects that are related to school studies, and which offer ample opportunity for vital problem work in arithmetic.

6. *Abstract problems dealing with relationship among numbers.* Very young children of superior intelligence frequently surprise their parents and teachers by their insight into number. This is seen in such children's questions as, "If 2 and 7 are 9, will 22 and 7 be 29?" Discussions like the

following frequently evoke earnest thought among superior pupils:

1. Multiply .3 by .2
   How can you tell that
   a. 6 is too large?
   b. .6 is too large?
   c. .06 is correct?
2. Divide 5.75 by .25
   Why do you think that
   a. .27 is too small?
   b. 2.7 is too small?
   c. 270 is too large?
   d. 27. is correct?
3. You have not yet learned how to divide by a decimal. You do not, therefore, know which of the following is correct.

$$\begin{array}{ccc} .21 & 2.1 & 21. \\ .14\overline{)2.94} & 1.4\overline{)2.94} & .14\overline{)2.94} \end{array}$$

   What rule for placing the decimal point does your textbook give? Test it by the above examples. Which is incorrect? Learn the rule. Make up other examples in which you must divide by a decimal. How will you test each example?

Compare this mode of learning the division of a decimal with the more orthodox method in which the child is led—the more skilful the leading, the more valuable the teaching, it is assumed. Here the child is challenged at every turn in the learning situation. But we must not overlook our basic assumption that only the most gifted children can acquire arithmetic skill by such successive intellectual challenges.

Steps in solving a problem. Orthodox advice on the solution of problems counsels that we find what is given, ascertain what is required, and then discover the relation between them. In the final analysis the child is really told that the best way to solve a problem is to solve it. A more definite procedure must be formulated if children are to develop a mode of thinking an arithmetical problem through to a reasonable solution. While there is no assured method of thinking any problem to a correct conclusion, there are modes of attack that promise success and others that lead to failure. We must now consider, in detail, those steps which are designed to develop intelligent planning and creative thinking in solving arithmetical problems. Five definite steps may be identified.

1. Grasping the situation.
2. Ascertaining (*a*) what is to be sought, the unknown and (*b*) what is given, the known.
3. Planning the solution.
4. Carrying out the plan.
5. Checking the answer.

1. **Grasping the situation.** Inability to solve a problem may frequently be traced to failure to understand its background. City children may not be able to visualize the path around the garden, the pickets in the fence, or the number of running feet in a hedge that surrounds an estate. Many children do not understand the reason for deliberately asking, in catalogs, for a higher price than one really expects to receive. Unless they can imagine what is actually occurring in a problem they cannot take the first step towards its solution.

The language of arithmetic may conceal the thought. We employ such terms as *list price, net price, catalog price, discount for cash, credit and agent,* forgetting that not all children understand them. Many of them have very limited vocabularies, and therefore do not understand even the non-technical language in which the problem is expressed. Those who are still learning to read may easily misread the conditions of a problem.

Children must be asked to tell, in their own words, the conditions of the problem; they may be called upon to dramatize the successive steps. Words and phrases beyond their expressional vocabularies must be clarified. Pupils must be encouraged to tell the teacher the words or the facts that they do not understand.

It seems wise to insist on two readings of the problem: first, to get the situation; second, to get the numbers or quantities. In reading the text the reading span is comparatively large; in reading numbers the span is comparatively small. The first reading, therefore, must be concerned with the plot in the problem, the second with its numbers. To be sure, this initial step in problem-solving takes time, but it is absolutely essential if the process of solution is to be thoroughly rational.

An investigation revealed that exercises in which emphasis was placed on getting the meaning of problems, aside from their arithmetical implications, produced decided improvement in ability to solve problems.[7] In developing a method of problem-solving, Rousseau's aphorism seems to hold true: to gain time, spend time.

2. Ascertaining the unknown and the known elements. When the situation presented in the problem is understood the solution should be attempted. The first step is to set forth in skeleton form what is sought and what is known, thus:

> A man must ship 720 packages. If he sends them by parcel post, the cost per package is 12 cents: To get these packages to the post-office the man must pay a truckman $12.50. If he sends them through the express company, he will have to pay 25 cents for each package and nothing more because the express company sends its wagon direct to the shipper. Which do you advise, shipping by parcel post or by express? How much will the man save by following your advice?
>
> *Find:* (1) Which is better; (2) by how much.
> *Given:* Cost per package by parcel post; cost of carting to post-office; cost per package by express.

This procedure is absolutely necessary to insure work that is planned and that moves toward a definite goal. We commonly find children plunging into a problem without a preliminary analysis of what is given and without a clear realization of what is sought. In the midst of the solution they occasionally become aware of an absence of aim, and confusion follows. To discourage thinking with the end of the pencil, teachers often tell their children, "Put down your pencils and think." The class sits impatiently a moment until the signal to begin is given. But the command, think, is vague and induces a feeling of helplessness. Children must be told what to think about or how to think. To require them to set forth the elements of the problem under the words *Find* and *Given* leads to an analysis of the problem and to a familiarity with its conditions, as well as to a recognition of the end to be attained. This is real thinking. Children should be permitted to proceed with their work only after completing this preliminary step. This procedure should be continued until it becomes habituated.

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[7] Estaline Wilson, "Improving the Ability to Read Arithmetic Problems," *Elementary School Journal*, Vol. 22, January, 1922, pp. 380-386.

3–4. Planning the solution and executing the plan. After the problem is clearly grasped, a plan must be formulated for attaining the correct answer. The steps should be indicated clearly in writing, before computation is begun. Let us assume that the following problem is to be solved; steps 2 and 3 would then appear as here set forth.

Mr. L. took a position in New York City at a salary of $2750 a year. After settling in his new home he had $3600 left in the bank, which pays 4% interest semiannually. What is his total income at the end of the year?

Find: Total income.
Given: Salary, $2750; principal, $3600; int. 4%, semiannually.
Plan: 1. Find int. for 6 mo.
2. Find int. for next 6 mo.
3. Find total income.

It may be objected that writing out a plan adds appreciably to the time required for solution, and that in business no such procedure is required. We must remember that so much penmanship need not be required in all grades. After children have acquired the habit of examining a problem to find known and unknown elements and of planning the solution, all written evidence of these mental operations may be reduced. If the form of work suggested above were followed out in the fifth and the sixth year of the elementary course, children in the seventh year would no longer precipitate confusion by rushing into a problem without a survey of its conditions and a tentative plan for its solution. The pupil in grades below the seventh year is not yet ready to meet the demands of business. He is engaged in acquiring foundational habits that will insure effective adjustment to life's demands. There is no virtue in continuing to write out the plan after one has firmly established the habit of planning his mode of solving problems.

Methods of solving problems. Various methods must be taught for solving problems; each has its appropriate place in the elementary-school course. We shall present them in tabular form before discussing their peculiar values:

1. The graphic method
2. The method of analysis
    a. Unitary analysis
    b. Ratio analysis
    c. Algebraic analysis
    d. Analysis through proportion
3. The rule or formula

In the intermediate grades (5 and 6) these methods of solution are kept so distinct that each maintains its identity. But in the upper grades (7, 8, and 9) children do not "resort to the conventional methods of analysis."[8] As a rule superior performance is attained through graphic modes of solution.

The graphic method. When the type of problem is new and the relationships do not suggest themselves readily, a graphic representation of them is helpful. Let us consider two illustrations:

> A school allows Grade 6A a plot in its garden measuring 9 × 18 ft. A path 1 foot wide is run through the middle of the plot, parallel to the 18 ft. sides. Each pair of children is given 8 sq. ft. of ground to plant. How many small plots were thus made and how many children planted seeds in this class garden?
>
> A newspaper, in telling of the campaign to raise funds for a new hospital, says that the committee in charge of the plan is encouraged because, "Contributions now total $32 000 and thus ⅝ of the money is assured . . ." How much money is needed for the new hospital?

In the first problem we find a set of facts which the child cannot synthesize unless he expresses them in a drawing. He is encouraged to translate the facts, one by one, into a map of the plot. First the large plot is represented by a rectangle, 9×18. A path, one unit wide, is then run through the plot as indicated in the problem. The next step is to lay out promised gardens. The plan in the diagram is the most obvious and the small gardens are drawn in half the large plot. The rest is simple enough.

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[8] John R. Clark and E. Leona Vincent, "A Comparison of Two Methods of Arithmetic Problem Analysis," *Mathematics Teacher,* Vol. 18, April, 1925, pp. 226–233.

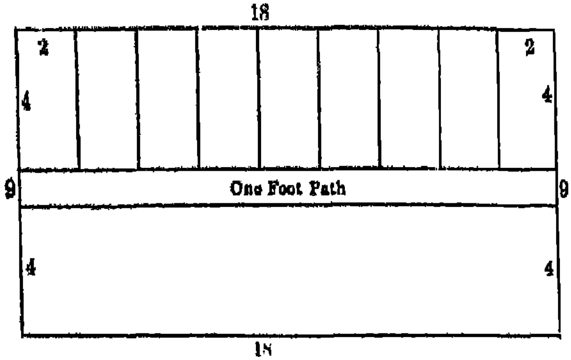

FIG. 20.

In the second problem the relationship among the quantities may be difficult to grasp. Here, again, the graphic representation helps the child. All the money needed is represented by a rectangle. Since ⅔ of the money needed is collected, the rectangle is divided into thirds and two of these parts are set off from the remaining one. The child sees that if ⅔ of the whole diagram represents $32 000, then ⅓ repre-

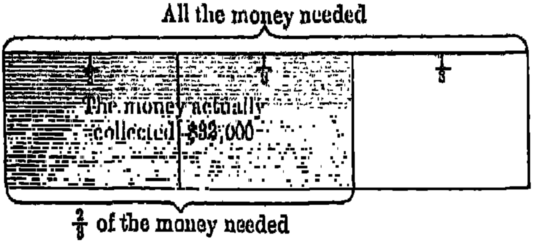
All the money needed
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
The money actually collected $32,000
$\frac{2}{3}$ of the money needed

FIG. 21.

sents $16 000. The conclusion is almost inescapable—the 3 thirds are $16 000×3 or $48 000.[9]

The method of solution by analysis. When the type of problem is new, concrete aids are psychologically justified because concepts or general ideas develop from perceptional or sense experiences. As the type is mastered, less and less reliance must be placed on objective material. The child learns to reason as follows:

> ⅔ of the money needed has been collected
> $32 000 has been collected
> ⅔ of the money collected is, therefore, $32 000
> ⅓ of the money collected is, therefore, $16 000
> $\frac{3}{3}$ of the money collected is, therefore, $16 000 × 3 or $48 000.

Young children are likely to think: "⅔ of the money is $32 000, then ⅓ of it is ⅓ of $32 000. To meet this difficulty, refer to the diagram again and lead pupils to deduce:

> if 2/parts of the diagram represent $32 000, then 1/part of the diagram represents ½ of $32 000

But each part is a third; then we may say,

> if 2/thirds of the diagram represent $32 000, then 1/third of the diagram represents ½ of $32 000.

It is obvious that 1/third may be written in a simpler form, ⅓, hence

> if ⅔ of the diagram represents $32 000, then ⅓ of the diagram represents ½ of $32 000.

The fraction in the problem is now changed to ⅖, $\frac{2}{9}$, $\frac{2}{7}$ and the reasoning repeated. The children soon see that the parts may be designated as two-thirds or ⅔, or 2/parts or 2/boxes, or 2/sections; that the number of parts, not the name of the parts is vital in finding the value of one of the parts. In this way the gap between graphic and analytical solution is bridged.

In teaching problems in percentage, the method of analysis should be used unless the type is new. Consider the example: A merchant who failed in his business, can pay only 15% of his debts. Mr. B. receives $2550 from this merchant. How much was due Mr. B.?

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[9] This graphic analysis is not a concrete presentation but a diagrammatic representation, as was explained earlier (page 127). Although not concrete, it is more effective than a verbal analysis of the conditions in the problem.

The type of problem is not new; it is the same as the fractional problem in which ⅔ of the money needed for the new hospital was collected. It is reasonable, therefore, to expect children to learn the following mode of analytical reasoning:

Mr. B received 15% of the money due him.
Mr. B received $2550.
Therefore, 15% of the money due Mr. B = $2550 (we wish to know 100%).
1% of the money due Mr. B. = $\frac{1}{15}$ of $2550 = $170.
All, or 100%, of the money due Mr. B. = $170 × 100 = $17 000.

Thus far we have illustrated analytical solutions by unitary analysis. The whole is found in terms of *one*-third, *one*-ninth, or one-per cent. We must not overlook analytical solutions by the ratio method. Let us consider the following examples and contrast the two forms of analysis:

A druggist paid $2.31 for ⅞ of a pound of a drug. What will 6⅛ pounds of this drug cost him?

Unitary Analysis	*Ratio Analysis*
⅞ lb. cost $2.31 ⅛ lb. cost $\frac{1}{7}$ of $2.31 or $.33 $\frac{8}{8}$ lb. cost $.33 × 8 = $2.64 6⅛ lb. or $\frac{49}{8}$ cost, $\overset{33}{\cancel{2.64}} \times \frac{49}{\cancel{8}}$ or $16.17	6⅛ lb. or $\frac{49}{8}$ lb. = 7 × ⅞ lb. 6⅛ lb. will cost 7 × $2.31 or $16.17.

How far will a train run in 4 hours and 40 minutes if it covered a distance of 70 miles in 2 hours and 20 minutes?

2 hr. and 20 min. = 2⅓ hrs. or $\frac{7}{3}$ hr. In $\frac{7}{3}$ h. train ran 70 mi. In ⅓ hr. train ran $\frac{1}{7}$ of 70 miles or 10 miles In $\frac{3}{3}$ hr. or 1 hr. train ran 10 × 3 miles or 30 miles In 4 hr. and 40 min. or 4⅔ or $\frac{14}{3}$ hr. train will run 30 mi. × $\frac{14}{3}$ or 140 miles.	4 hr. and 40 min. is twice 2 hr. and 20 minutes Therefore train will run 70 mi. × 2 or 140 mi.

It is obvious that children must be taught to examine the quantities given in the hope of finding factor-multiple relationships among them. While the method of ratio analysis is not universally applicable, to neglect it and solve uniformly by unitary analysis involves serious waste.

Analysis in terms of algebraic equations. Simple algebraic solutions can be learned by children in the fifth and sixth years despite the fact that no other algebraic operations are taught them. Let us compare the two forms of solution to the following problem.

Goods must not be sold for less than \$28.50 a gross after the customary 5% discount is allowed to shopkeepers. What price shall I mark them in the catalog? (C.P. denotes catalog price or list price.)

Usual Analysis	*Algebraic Solution*
C.P. = $20/20$	x denotes C.P.
Dis. = $1/20$ of C.P.	$x - 1/20$ denotes S.P.
$19/20$ of C.P. = final S.P.	\$28.50 is S.P.
\$28.50 = final S.P.	$x - 1/20 = \$28.50$
$19/20$ of C.P. = \$28.50	$20x - x = 570$
$1/20$ of C.P. = $1/19$ of \$28.50 = \$1.50	$19x = 570$
$\frac{20}{20}$ of C.P. = \$1.50 × 20 = \$30.	$x = \frac{570}{19} = 30.$ Ans. is \$30

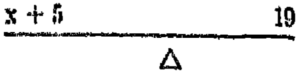
In the final analysis the only significant characteristic of the algebraic solution is the use of x in place of the letters C.P. Had the problem called for the finding of the commission or the profit or the cost, the unknown element would have been designated in the first form of solution by the abbreviation com. (commission), or g (gain), or C (Cost), respectively. In the algebraic method all unknown elements are uniformly represented by x. The algebraic form, it is argued, simplifies the solution through uniform terminology and the elimination of awkward expressions.

Only three types of algebraic equations need be taught to give the child a real measure of help in solving arithmetical problems. These are set forth below with simple indications of methods of presenting them.

Case I. Form $x + 5 = 19$, find x.

Method:

1. Introduce through practical problem.
2. Elicit equation form $x + 5 = 19$
3. Explain by device of a balance

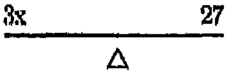


   a. Elicit: taking equal amounts from the two sides of the balance will not disturb the balance.
   b. Elicit: $x + 5 - 5 = 19 - 5$
   $x = 19 - 5$ or 14
   c. Elicit: $+ 5$ on left side of equation becomes $- 5$ on right side.
4. Introduce through a problem, the form $x - 5 = 22$
   a. By device of balance elicit:
   $x - 5 + 5 = 22 + 5$
   $x = 22 + 5$ or 27
   b. Elicit $- 5$ on left side of equation became $+ 5$ on right side.
5. Generalization: Law governing transfer of quantities in simple equations.

Case II. Form $3x = 27$

Method:

1. Introduce through a problem.
2. Reduce to equation form, $3x = 27$.
3. Use device of balance.



4. Elicit: dividing each side by 3 does not disturb the balance of the two sides.
5. Elicit: $3x \div 3 = 27 \div 3$
   $x = 27 \div 3$ or $^{27}/_{3}$.
6. Multiply instances.
7. Formulation of law for finding x in equations like $19x =$ \$570.

Case III. Form $x - \frac{19}{20}x = \$28.50$

Method:

Teach how to clear fractions through least common denominators.

Practice until habit is developed.

Analysis by ratio and proportion. Solution by proportion is not usually advocated for many reasons: (1) Proportion was discredited by the impossible problems that were frequently assigned for solution by this method. Examples of the following type were the rule rather than the exception: "If 42 men build a wall 52 feet long, 20 feet high, 3 feet thick, in 27 days, how long will the wall be if 21 men are set to work for 54 days on a wall 15 feet high and 2 feet thick?" The demand for the socialization of arithmetic brought an elimination of these puzzles and a diminished use of proportion. (2) The algebraic solution advocated by many teachers offers a satisfactory substitute for proportion in many types of problems. (3) With the popularization of the ratio mode of analysis as opposed to the unitary analysis, proportion was found too cumbersome in many combinations. The truth of this is illustrated in the following problem:

A freight train running at the rate of 16 miles an hour takes 72 minutes between two towns. How long will the special milk train take for this trip if it runs at a speed of 48 miles an hour?

By ratio and proportion:

16 miles : 48 miles = ? min. : 72 min.

$$16 \times 72 \div 48 = \frac{\overset{1}{\cancel{16}} \times \overset{24}{\cancel{72}}}{\underset{3}{\cancel{48}}} = 24 \text{ minutes}$$

By ratio analysis:

Speed of 48 miles = 3 times speed of 16 miles.

Time necessary hence ⅓ of 72 minutes or 24 minutes.

These reasons account for the reduced position now given the subject of proportion in courses of study in arithmetic.

There are so many helpful uses of proportion that many authorities urge that it be restored to its former place. They argue that many common relations are expressed in ratio forms. Thus we speak of a 3 to 1 chance, of the naval ratio of 5:5:3, of the free coinage of silver at a ratio of 16 to 1, of odds that are 5 to 2, and the like. How can children carry out a drawing to scale without applying proportion? In certain problems solution by proportion insures the simplest procedure. A few illustrations may give meaning to these assertions.

When a commodity is selling at $24 a ton, what should be paid for 1750 pounds?
Ratio Analysis: 1750 is $\frac{7}{8}$ of a ton, but only a very few children would perceive this relation.
Solution: $\frac{7}{8}$ of $24 = $21.
Unitary Analysis: Most children solved this example in terms of the cost of 1 lb.
$\frac{1}{2000}$ of $24 × 1750 = Cost
Proportion: $24 : $? = 2000 : 1750

$$\frac{\overset{\overset{7}{\cancel{35}}}{\cancel{1750}} \times \overset{3}{\cancel{24}}}{\underset{\underset{\underset{1}{\cancel{8}}}{\cancel{40}}}{\cancel{2000}}} = \$21$$

If $1\frac{1}{2}$ inches on a map represents a distance of 360 miles, how great a distance is represented by $3\frac{1}{8}$ inches on the same scale?
Ratio Analysis: $3\frac{1}{8}$ is not a multiple of $1\frac{1}{2}$ and no obvious ratio can be seen.
Unitary Analysis: $\frac{3}{2}$ in. represent 360 miles
$\frac{1}{2}$ in. represents $\frac{1}{3}$ of 360 miles or 120 miles
$\frac{2}{2}$ in. represents $\frac{2}{3}$ of 360 miles or 240 miles
$3\frac{1}{8}$ in. represents 240 miles × $\frac{25}{8}$ or 750 miles
Proportion: $1\frac{1}{2}$ in. : $3\frac{1}{8}$ in. = 360 miles : ? miles.

$$\frac{25}{8} \times 360 \div \frac{3}{2} =$$

$$\frac{25}{\underset{\cancel{4}}{\cancel{8}}} \times \overset{\overset{30}{\cancel{120}}}{\cancel{360}} \times \frac{2}{\cancel{3}} = 750 \text{ mi.}$$

In these two problems proportion involves less analysis and leads directly to the answer.

To teach ratio and proportion it is necessary to give the child a rich concept of the ratio idea. This is best done through use of ratios rather than through definitions and formal explanations. The following procedure, it is hoped, will be found helpful:

a. Concept of ratio. Practice should be given in examples of the type, "2 is what part of 4" and children's answers should be tabulated as indicated.

Teacher asks:	*Pupil answers:*	*Teacher tells:*
2 is what part of 4?	$\frac{1}{2}$	The ratio of 2 to 4 is $\frac{1}{2}$; this may be written 2:4 or $\frac{2}{4}$ or $\frac{1}{2}$.
3 is what part of 6?	$\frac{1}{2}$	The ratio of 3 to 6 is $\frac{1}{2}$; this may be written 3:6 or $\frac{3}{6}$ or $\frac{1}{2}$.
2 is what part of 6?	$\frac{1}{3}$	The ratio of 2 to 6 is $\frac{1}{3}$; this may be written 2:6 or $\frac{2}{6}$ or $\frac{1}{3}$.
3 is what part of 9?	$\frac{1}{3}$	The ratio of 3 to 9 is $\frac{1}{3}$; this may be written 3:9 or $\frac{3}{9}$ or $\frac{1}{3}$.

After sufficient practice is given in this initial step, the children fill out the entire line for "6 is what part of 9," "6 is what part of 8," etc. The meaning of the term ratio, the relation of one quantity to another, is now elicited.

b. Ratio as a fraction. The next step emphasizes the fractional equivalent of every ratio. Children write $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{9}{15}$ as ratios and 5:8, 7:16, 9:12 as fractions. This step can readily be deduced from the introductory step outlined above.

c. Introduce the proportion. To introduce the concept *proportion,* two ratios in fractional forms are used.

Teacher gives:	*Children are required to write:*
$\frac{4}{6}=\frac{2}{3}$	4:6=2:3
$\frac{9}{15}=\frac{3}{5}$	9:15=3:5

d. Finding fourth term, knowing other three. The teacher gives 3:5=9:? and asks children to give the fourth term. If they hesitate, recall the fractional equivalents $\frac{3}{5}={}^{9}/?$; it is evident that the numerator of the first fraction must be multiplied by 3 to obtain 9, the numerator of the second fraction; therefore the denominator, 5, of the first fraction, must also be multiplied by 3; hence 15 is obtained. This sequence is applied to a number of examples and the resulting equivalents are arranged thus:

Teacher gives:	*Children supply:*	*Analysis shows:*
3:5=9:?	3/5=9/?	9÷3 (to obtain 3, the common multiplier) × 5 9÷3×5 or 9×5÷3 which gives the fourth term.
6:7=12:?	6/7=12/?	12÷6 (to obtain 2, the common multiplier) × 7 12÷6×7=12×7÷6 which gives the fourth term.
4:5=12:?	4/5=12/?	12÷4 (to obtain 3, the common multiplier) × 5 12÷4×5=12×5÷4 which gives the fourth term.
3:4=15:?	3/4=15/?	15÷3 (to obtain 5, the common multiplier) × 4 15÷3×4=15×4÷3 which gives the fourth term.

The teacher now introduces the technical terms *means*, *extremes*, and *proportion*. The analysis of each case shows that the product of the extremes equals the product of the means; that the product of the means divided by one extreme gives the other extreme. Many practical problems should now be solved by proportion.

e. The unknown term is one of the means. Children who know how to find the fourth term of the proportion 3:5=15:? may not have enough insight into number relations to be able to complete the proportion 3:5=?:15. This new form is necessary in problems of the type, "If 72 men working 10 hours a day complete a given work, how many men will probably be needed under the law limiting the hours of labor to 8 a day?" Here the proportion would be inverse, 10 hrs.:8 :: ?:72, because more men are needed as the working day is reduced. The mode of explanations and analysis is similar to that suggested in step (*d*) except that the law of proportion, the famous *rule of three,* is enriched by the statement, *the product of the extremes divided by one mean gives the other mean.* Practical problems in inverse and direct proportion are now given.

(*f*) Ratio of difference or unlike units. This step is simple, for in the ratio of 2 yards to 2 feet it is merely necessary to reduce the quantities to the same units. The ratio now becomes 2 yards to ⅔ yard or 6 feet to 2 feet.

Cautions in the analysis of problems. We have seen that problems may be analyzed by unitary analysis, by ratio method, by algebraic equations, and by proportion—each mode involving a measure of cancellation. There are important cautions that must be exercised if the analysis of problems is to be rendered effective.

Importance of correct algorisms. Looseness of mathematical language betrays looseness of thought. It was shown in an earlier chapter that algorisms of the form of ⅗=cost and ⅕=⅓ of cost must not be permitted. But it is not wise to become too meticulous about form. To condemn the cancellation form, ⅔×$5.70×10⁄7, because there is here implied a process of multiplication in which one multiplier is concrete, is to insist on futile refinements.

Only when verbal analyses of problems express the thought processes in the mind of the child do they train in systematic and sustained thinking. Usually verbal analyses are fraught with danger, because they assume that thought accompanies the child's words. When the pupil recites, "If ⅗ of the money is $27.90, ⅕ of the money is ⅓ of $27.90 and 5⁄5 of the money is ⅓ of $27.90×5 or 5⁄3 of $27.90," he may be repeating the teacher's *therefore's, because's,* and *since's.*

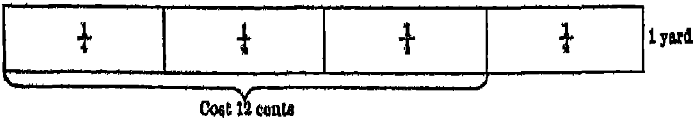
Verbal analyses should be discontinued just as soon as the process is understood by the pupils. It is well to encourage children to express their ideas in their own language, despite the fact that the set forms are better, linguistically and arithmetically. No verbal analysis of problems should be required of children in the first four grades. In all grades the diagrammatic expression of the relationship in a new problem should be preferred to the verbal analysis.

Solving problems by rule or formula. In the tabulation of methods of solving problems we find (1) the graphic method, (2) the method of analysis with its four distinct types, and (3) the use of a rule or formula. The development of the inductive method brought the use of rules into bad repute. Teachers had so long given lessons that began with laws and formulas, and had so frequently presented symbols before thought, that solution of problems by rule fell into disrepute.

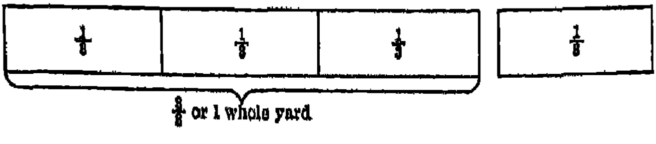
To-day, we believe that solution by rule or formula is justified if the rule is taught after the child has had rich perceptional experiences. The solution of a new type of problem should be graphic. After the children become familiar with this type, a form of solution by analysis is employed. When the mode of thinking is reduced to habit, solution by rule is taught.

Let us assume that we are to teach how to find the cost of one, knowing the cost of a fractional part of it. The examples are,—

If ¾ of a yard cost 12 cents, what will a yard cost?
What will I pay for a yard if 1⅓ yards cost 16 cents?
First we insist on the graphic solution, thus,



Each quarter-yard must cost 4 cents if 3 of the quarters cost 12 cents.
The whole yard or 4/4 must cost 4 times as much as 1 quarter or 16 cents.
Similarly,



Four of these thirds cost 16 cents, 1 third costs 4 cents, and 3 thirds, 12 cents.

The method of analysis follows in the form presented earlier.

If $\frac{3}{4}$ yard cost 12 cents
$\frac{1}{4}$ yard cost $\frac{1}{3}$ of 12 cents or 4 cents
$\frac{4}{4}$ yard cost four times as much or 4 cents $\times$ 4 or 16¢

In the second example we proceed by analogous reasoning:

$1\frac{1}{3}$ yards or $\frac{4}{3}$ yd. cost 16¢
$\frac{1}{3}$ yd. costs $\frac{1}{4}$ of 16¢ or 4¢
$\frac{3}{3}$ yd. cost three times as much, $4¢ \times 3 = 12¢$

After many such problems are solved, the children are led to see that in the first example 12 cents is divided by 3 and multiplied by 4, thus: $12 \div 3 \times 4$; in the second 16 is divided by 4 and multiplied by 3, thus: $16 \div 4 \times 3$. What we get, therefore, is

$$\begin{array}{lll} 12 \div 3 \times 4 \quad \text{or} & \qquad & 16 \div 4 \times 3 \quad \text{or} \\ \downarrow \quad | \quad \downarrow & & \downarrow \quad | \quad \downarrow \\ \dfrac{12 \quad \downarrow \times 4}{3} \quad \text{or} & & \dfrac{16 \quad \downarrow \times 3}{4} \quad \text{or} \\ \dfrac{12 \times 4}{3} & & \dfrac{16 \times 3}{4} \end{array}$$

The cost of a fractional quantity is divided by the fractional quantity to find the cost of one. The rule is now merely a succinct statement or summary of the arithmetical operations performed in the long verbal analysis. Coming at the end of a semester of work, after graphic and analytical methods of solution have become thoroughly familiar, the rule is neither an arbitrary statement nor an empty formula. It is rather a rich and meaningful summary of actual experiences gained in solving a particular type of problem.

It is commonly assumed that such arithmetical facts as $5 \times 9 = 45$, $6 \div \frac{1}{6} = 36$, and $6 \div 1 = 6$ are learned through imitation, but that problems are solved in some mysterious way through the power to think. This is decidedly untrue, because our thinking is in terms of patterns learned as definitely as $6 \div 1 = 6$. We learn through a pattern imposed upon us in youth by textbook or teacher, that when we are given *the cost* *of many* (c) and *the number of things* (b) and are told to find *the cost of one* (a), we divide c by b and obtain a, thus $\frac{c}{b}=a$. After practice in solving problems of this type, we alight upon the correct mode of solution almost as mechanically as we divide 360 by 6. This is as it should be, for these problems refer to a typical situation in life. In typical situations of daily occurrence the solution should become habituated very soon after the initial rationalization.

5. Checking the answer. We suggested that children be taught to take five steps in solving problems: (1) grasping the situation; (2) ascertaining what is to be found and what is given; (3) planning the solution; (4) carrying out the plan. We come now to the fifth step—checking the answer. It is easy to make clear to children the great importance of absolute correctness of answers. They must be taught to employ at least one of four procedures in checking answers to problems:

a. Working problems backward.

Illustration: A local newspaper reports that business conditions are improving and that 15% of those unemployed found work during the month. Later in the report we are told that 360 found employment that month. How many were unemployed at the beginning of that month?

Answer: 2400 unemployed
Check: 15% of 2400 = 360

b. Working the problem by another plan.

Illustration: A man who had $10.00 made the following purchases: 2 shirts @ $1.15 each; 3 ties @ $.35 each; 1 dozen handkerchiefs @ 3 for $.25. What change did he receive?

First solution:

2 shirts	@	$1.15	cost	$2.30	$10.00
3 ties	@	.35	cost	1.05	−4.35
12 hdkfs.	@	4 for .25	cost	1.00	$5.65, change
				$4.35	

Second solution or check:

$10.00 − $2.30 = $7.70
7.70 − 1.05 = 6.65
6.65 − 1.00 = 5.65, change

c. Working each step again.

When neither of the first two methods suggested above seem feasible, the problem should be gone over, step by step. If the second answer agrees with the first, the chances of its correctness are increased. Strictly speaking, this is not a real check; it is an active challenge of the answer.

d. Comparing the final answer with the approximate answer.

Illustration: Mr. Jones keeps a record of the service his tires give him. Here it is:

Tire	No. 1	18 200 miles
	No. 2	17 500 miles
	No. 3	13 500 miles
	No. 4	14 050 miles

His friend urges him to buy the kind of tire he uses and tells him, "My tires average 15 000 miles." Mr. Jones does not know whether to change. What do his tires average?

Approximation: rapid addition of the mileage, omitting such numbers as 50 and 200, gives 63 000. Dividing 60 000 by 4 gives 15 000. The approximate answer is there, "well over 15 000 but under 16 000 miles."

True computation:

Total mileage, 4 tires 63 250 miles
63 250 miles ÷ 4 = 15 812.5 miles average

Comparison of the Answers: 15 812.5 agrees with the approximate answer.

Here, too, we have not an absolute proof, but an assurance that the answer is not ridiculously incorrect. The reader is referred to an earlier chapter in which checking answers is treated with greater fullness.

The improvement of problem work. *Analytical and diagnostic test of pupil ability.* Early in the work of the grade, teachers must determine through reliable tests, each pupil's

a. level of general intelligence;
b. reading ability;

c. mastery of arithmetic skills in the fundamental operations;
d. attainment in problem-solving.

Where exceptional deficiencies are found in *b* and *c*, appropriate remedial practice must be provided. Our earlier chapters indicate the corrective drills in fundamental operations. Obviously the child of limited intelligence will make little progress in problem-solving. Such children should be required to solve only those problems that follow easily recognized types. Studies show that ability in problem-solving correlates highly with verbal intelligence, mastery of fundamental operations in arithmetic, and comprehension of reading material. The diagnostic testing here suggested is designed to reveal both remediable and non-remediable causes of inability to solve arithmetic problems.

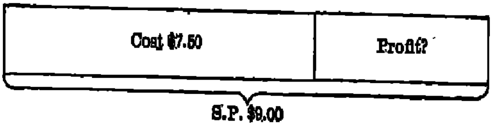
Careful scrutiny of textbook problems. Teachers must not accept blindly textbook material in the form in which it is offered. The problems in the book frequently need revision. Their language may not be the most appropriate for a given class. A single word may obscure the situation presented in a problem. To discover these interfering words children may be required to state the problem as they understand it. At times the teacher may restate the problem substituting a happier phrase for the one in the book. Often a few supplementary problems must be interpolated to bridge the steps in too abrupt a gradation. Not infrequently a problem assumes a knowledge of commercial or industrial practices that young people do not possess. At all times teachers must edit the textbook problems before submitting them to their classes for solution.

Encourage visualization of the conditions of a problem. Every aid should be given to children to visualize the conditions of the problem. An earlier suggestion, that children should express graphically the relationship among the quantities in a problem, now assumes added importance. Such practice as the following gives children facility in representing by diagram the conditions of a problem.

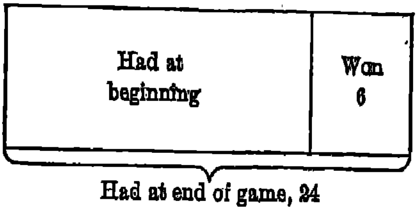
The assignment: show the following by diagram

The assignment carried out

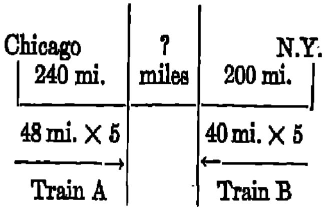
1. Goods cost $7.50 are sold for $9.00. Find the profit.



2. William, who had 24 marbles at the end of a game, was heard to boast that he won six in the last game. How many did he have at the beginning of the game?



3. Train A leaves Chicago and averages 48 miles an hour. Train B leaves N. Y. at the same time and averages 40 miles an hour. What other facts must you know to find out how far apart these trains are at the end of 5 hours?



The diagram tends to show more clearly that the distance between New York and Chicago must be known in order to find out how far apart the two trains are at the end of 5 hours.

Frequently the relationships are not seen by immature pupils until they substitute small numbers for the larger and more natural numbers of the problem. With increasing experience children grow less dependent on the actual diagram and find themselves able to visualize the relationships.

PROBLEMS FOR STUDY AND DISCUSSION

1. The text lists six factors that determine ability to solve arithmetic problems.
   (*a*) Give for each one or more illustrations.
   (*b*) How would you determine which of these factors are probably responsible for the failure of a particular child to solve arithmetic problems?

Apply your procedure to a real child if possible; if not, to an imaginary pupil.

2. Invent the following problems and examine each critically before submitting it for class criticism:
   (*a*) Three problems designed to familiarize the child with business or social practices.
   (*b*) Two problems arising in home making experiences.
   (*c*) Three problems based on recreational experiences.

3. Consider the problem: 2½ yards cost $7.50; what will 3¼ yards cost? Write four problems of the same type, making such changes in language and arrangement of the quantities as will keep them from being stereotyped.

4. Examine problems offered in a textbook to discover a basis for gradation. Do you approve of this principle of gradation? Give reasons.

5. With references to each type of problem presented in this chapter, give:
   (*a*) Three illustrations of your own invention.
   (*b*) Its specific values in training to think in arithmetical situations.
   (*c*) Its limitations—When should it be used? How frequently? With what kind of pupil?

6. A teacher complains: Children tabulate the elements of a problem under the headings of *Given* and *Find,* and list the steps in their plan after they have obtained the answer. They do this merely to obey the arbitrary demands of the school. This shows the requirement of all this planning does the children no good. Do you agree with the teacher? How can her position be challenged?

7. Cause of deficiency in problem-solving may be remediable or non-remediable. List the causes under each of these two headings. How would you deal with each cause in the first group?

8. "Even in problem-solving children think by a method acquired imitatively." Plan a discussion of this idea and enrich the exposition by many illustrations. Make your own position clear.

9. "The problems which are: (*a*) most often questions about situations or facts actually presented to the senses, (*b*) less often questions which a person puts to himself in connection with past or future plans, and (*c*) least often questions put to him in words by another." (*Contemporary Guide in the Teaching of Arithmetic*, p. 93.)

What practical application can be made of this statement in teaching problem-solving in arithmetic?

10. "The entire question of reasoning and the reasoning problem in arithmetic has been very inadequately investigated."

List questions and problems the answers to which await adequate investigation.

11. Examine one diagnostic test in problem-solving. If possible, administer this test. What facts concerning pupil needs are revealed? What remedial measures do you suggest?

12. Devise a test on the contents of this chapter designed for your classmates. Only Part I of the test should consist of new-type questions. What will the content of Part II be? Justify the organization, distribution of values, and the test elements.

APPENDIX I

A SELECTED, CLASSIFIED, AND ANNOTATED BIBLIOGRAPHY

In the compilation of this bibliography *completeness was not sought.* The aim is rather to suggest materials which are comparatively not inaccessible and which have played an important part in the development of our present knowledge of the psychology and pedagogy of arithmetic. Only those primary sources have been included that either make a significant contribution or illustrate effective research techniques. Numerous other bibliographical references will be found in foot-notes throughout the book. Not all of these are here repeated. References to Buswell, Monroe, and others who have formulated reasonably complete bibliographies are included below.

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——, and HAWKINSON, "The Optimum Order of Arrangement of Items in a Diagnostic Test," *Elementary School Journal,* Vol. 34, No. 5 (January, 1934), pp. 351–357. In diagnostic tests designed for individual diagnoses, the four test examples of each type should be arranged together in one row. The test should be given during the time when the pupils are learning the process.

BUSWELL, G. T. and JOHN, Lenore, "Diagnostic Studies in Arithmetic," *Supplementary Educational Monographs,* No. 30 (University of Chicago, 1926). Report of over 500 individual diagnoses. Difficult combinations are identified and children's habits of work in arithmetic are indicated.

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CLARK, J. R., and VINCENT, E. L., "A Study of the Effect of Checking upon Accuracy in Addition," *Mathematics Teacher,* Vol. 19, February, 1926, pp. 65–71. Children taught to check performed fewer examples but had greater number correct.

——, "A Comparison of Two Methods of Arithmetic Problem Analysis," *Mathematics Teacher,* Vol. 18, April, 1925, pp. 226–233.

IMMACULATA, Sister M., "A Critical Study of Arithmetic Reasoning Tests," *Educational Research Bulletins,* Vol. 2, No. 6 (Washington, D. C., Catholic University of America, 1927). "The results of this experiment, although based on the scores of a rather small number of pupils" (quoted from this study, p. 34) show that the Buckingham Tests are superior to others in use in 1926.

JOHN, L., "Difficulties in Solving Problems in Arithmetic," *Elementary School Journal,* Vol. 31, November, 1930, pp. 202–215. A study of errors in solving two-step problems in Grades 4–6 in the University Elementary School of the University of Chicago and in a Chicago Public School.

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Lazar, May, *Diagnostic and Remedial Work in Arithmetic Fundamentals for Intermediate Grades* (Bureau of Reference, Research, and Statistics, Board of Education of the City of New York, 1928). A very helpful diagnostic analysis with appropriate remedial suggestions and case study. An important contribution.

Lazarete, M. E., *The Development of Problem-Solving Ability in Arithmetic* (Toronto, Clarke, Irwin & Co., Ltd., 1933). Considered a major contribution to this field by Buswell.

Lutes and Samuelson, "A Method for Rating the Drill Provisions in Arithmetic Textbooks," *University of Iowa Monographs in Education*, First Series, No. 3 (1926). Some students regard this as the most valuable aid in determining the value of drill material.

Mitchell, Claude, "Problem Analysis and Problem-Solving in Arithmetic," *Elementary School Journal*, Vol. 32, February, 1932, pp. 464–465. Detailed analytical questions asked by the teacher on problems to facilitate solution by the children.

Monroe and Clark, "The Teacher's Responsibility for Devising Learning Exercises in Arithmetic," University of Illinois Bureau of Educational Research, *Bulletin* No. 31, Vol. 23, No. 41, 1926. A careful and helpful analysis is made of the problems in ten textbooks.

Morton, R. L., "An Analysis of Pupils' Errors in Fractions," *Journal of Educational Research*, Vol. 9, February, 1924, pp. 117–125. Excellent diagnostic study with indications for remedial measures.

Neulen, Leon, *Problem-Solving in Arithmetic* (Columbia University, 1931). The aim of the study is to ascertain grade placement of various written problems, depending on the number of steps involved.

Newland, T. E., "A Study of the Specific Illegibilities Found in the Writing of Arabic Numerals," *Journal of Educational Research*, Vol. 21, No. 3 (March, 1930), pp. 177–185. The extent of illegibility.

Norem, G. B., and Knight, F. B., "The Learning of the 100 Multiplication Combinations," in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education (Public School Publishing Co., 1930), Ch. vii. Seeks to determine the amount of practice needed for each of these combinations. The relative difficulty of the combinations differs from that indicated by Clapp.

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———, and Sharp, E. T., "Long Division vs. Short Division," *Journal of Educational Research,* Vol. 26, No. 1 (September, 1932), pp. 6–11. Study shows that children prefer long division and are more accurate in it. The author concludes that long division should be taught first and that short division be taught as a short cut.

Overman, James R., "An Experimental Study of Certain Factors Affecting Transfer of Training in Arithmetic," *Educational Psychology Monographs,* No. 29 (Warwick and York, 1931). An extended research, interesting for its technique rather than its conclusions. Transfer takes place "in useful amounts" and is quickened by the extent to which the methods of instruction help the child to generalize.

Pressey, L. C., and Elam, M. K., "The Fundamental Vocabulary of Elementary-School Arithmetic," *Elementary School Journal,* Vol. 33, September, 1932, pp. 46–50. A fundamental vocabulary for elementary arithmetic is presented. The authors derive their data from (*a*) four previous studies; (*b*) ratings of 100 grade-school teachers; (*c*) a study of social value of words.

Ruch, G. M., "Relative Difficulty of the 100 Muliplication Facts with Special Reference to Textbook Construction," *Elementary School Journal,* Vol. 32, No. 5 (January, 1932), pp. 369–377. Summary and comparison of the facts in the three studies: by Clapp, by Norem and Knight, and by Foulkes.

Stevens, B. A., "Problem-Solving in Arithmetic," *Journal of Educational Research,* Vol. 25, Nos. 4–5 (April and May, 1932), pp. 253–260. Ability in fundamental operations is more closely correlated with ability in problem-solving than is general reading ability. Do children solve problems by following type solutions or by "vital reasoning processes"?

Stretch, Lorna B., *The Relation of Problem-Solving Ability in Arithmetic to Comprehension in Reading,* Contributions to Education, No. 87 (George Peabody College for Teachers, 1931). Investigation shows the value of training for specific purposes.

Washburne and Osborne, "Solving Arithmetic Problems," *Elementary School Journal,* Vol. 27 (November and December, 1926, pp. 219–226 and 296–304. Three methods of teaching problem-solving are evaluated.

Washburne, Carleton, "Mental Age and the Arithmetic Curriculum," *Journal of Educational Research,* Vol. 23, No. 3 (March, 1931), pp. 210–231. Indicates the placement of arithmetic topics with relation to mental age. For each topic the minimum and the optimum mental age is indicated. Very suggestive.

WASHBURNE, Chas. G., "The Grade Placement of Arithmetic Topics," in the *Twenty-Ninth Yearbook* of the National Society for the Study of Education (Public School Publishing Co., 1930), Ch. xiii. The placement of arithmetic topics is determined in terms of mental level and arithmetic foundations of children. An interesting inquiry.

WILSON, Guy M., "New Standards in Arithmetic: A Controlled Experiment in Supervision," *Journal of Educational Research*, Vol. 22, No. 5, pp. 351–360. Results "suggest replacing time on drill in Grades 1 and 2 by . . . more fundamental preparation . . . through understanding and experience."

STANDARDIZED TESTS: SURVEY AND DIAGNOSTIC

Bibliographical data pertaining to selected survey or inventory and diagnostic tests will be found in chapter XIV, especially on pages 166–177. Brief statements of the characteristics of each of these tests will be found in tabular form on pages 176 and 177.

CORRECTIVE PRACTICE MATERIAL

BRUECKNER, ANDERSON, BANTING and MERTON, *Diagnostic Tests and Remedial Exercises* (John C. Winston Co., 1931). For each of Grades, 3-8, the authors offer survey and diagnostic tests with appropriate remedial practice material.

COURTIS, S. A., *Courtis Standard Practice Exercises in Arithmetic* (World Book Co., 1920). The operations with whole numbers are covered in forty-eight practice cards for all grades beginning with the fourth.

LENNES, N. J., *Lennes' Work, Drill, and Test Sheets* (Laidlaw Bros., 1924).

OSBURN, W. J., *Osburn's Corrective Exercises in the Fundamentals of Arithmetic* (Houghton Mifflin Co.).

RUCH, KNIGHT and STUDEBAKER, Arithmetic Work Books (Scott, Foresman & Co., 1926). Separate books for Grades 3–8. An invaluable supplementary aid in these grades.

THORNDIKE, E. L., *Thorndike Practice Books in Arithmetic* (Rand, McNally and Co., 1921).

APPENDIX II

SAMPLES OF STANDARDIZED TESTS AND RELATED MATERIAL

The Rice Tests[1]

Fourth Year

1. A man bought a lot of land for $1743, and built upon it a house costing $5482. He sold them both for $10 000. How much money did he make?

2. If a boy pays $2.83 for a hundred papers, and sells them at four cents apiece, how much money does he make?

3. If there were 4839 classrooms in New York City, and 47 children in each classroom, how many children would there be in the New York schools?

4. A man bought a farm for $16 575, paying $85 an acre. How many acres were there in the farm?

5. What will 24 quarts of cream cost at $1.20 a gallon?

6. A lady bought 4 pounds of coffee at 27 cents a pound, 16 pounds of flour at 4 cents a pound, 15 pounds of sugar at 6 cents a pound, and a basket of peaches for 95 cents. She handed the storekeeper a $10 note. How much change did she receive?

7. I have $9786. How much more must I have in order to be able to pay for a farm worth $17 225?

8. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much money do I make?

Sixth Year

1. If a boy pays $2.83 for a hundred papers, and sells them at 4 cents apiece, how much does he make?

2. What will 24 quarts of cream cost at $1.20 a gallon?

3. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much do I make?

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[1] Tests for the fifth and seventh years are not reproduced. For full reference, see page 167.

4. A flour merchant bought 1437 barrels of flour at $7 a barrel. He sold 900 of these barrels at $9 a barrel, and the remainder at $6 a barrel. How much did he make?

5. If a train runs 31⅔ miles an hour, how long will it take the train to run from Buffalo to Omaha, a distance of 1045 miles?

6. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

7. The salt water which was obtained from the bottom of a mine of rock salt contained .008 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

8. A gentleman gave away ⅐ of the books in his library, lent ⅙ of the remainder, and sold ⅕ of what was left. He then had 420 books remaining. How many had he at first?

Eighth Year

1. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

2. The salt water which was obtained from the bottom of a mine of rock salt contained 0.08 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

3. A gentleman gave away ⅐ of the books in his library, lent ⅙ of the remainder, and sold ⅕ of what was left. He then had 420 books remaining. How many had he at first?

4. A man sold 50 horses at $120.00 each. On one-half of them he made 20 per cent, and on the other half he lost 10 per cent. How much did he gain?

5. Sold steel at $27.60 a ton, with a profit of 15 per cent, and a total profit of $184.50. What quantity was sold?

6. A fruit dealer bought 300 apples at the rate of 5 for a cent, and 300 at 4 for a cent. He sold them all at the rate of 8 for 5 cents. What per cent did he gain on his investment?

7. The insurance on ⅔ of the value of a hotel and furniture cost $420.00. The rate being 70 cents on $100.00, what was the value of the property?

8. Gunpowder is composed of niter 15 parts, charcoal 3 parts, and sulphur 2 parts. How much of each in 360 pounds of powder?

STONE'S TEST[2]

The main purpose of the reasoning test is the determination of the ability of 6A children to reason in arithmetic. To this end, the problems as selected and arranged are meant to embody the following conditions:

1. Situations equally concrete to all 6A children.
2. Graduated difficulties
    a. As to arithmetical thinking,
    b. As to familiarity with the situation presented.
3. The omission of
    a. Large numbers,
    b. Particular memory requirement,
    c. Catch problems,
    d. All subject matter except whole numbers, fractions, and United States money.

The test is purposely so long that only very rarely will any pupil fully complete it in the fifteen-minute limit.

The Test in Fundamental Operations

Work as many of these problems as you have time for; work them in order as numbered.

1. Add
2375
4052
6354
260
5041
1543

2. Multiply 3265 by 20.
3. Divide 3328 by 64.
4. Add
596
428
94
75
302
645
984
897

5. Multiply 768 by 604.
6. Divide 1918962 by 543.
7. Add
4695
872
7948
6786
567
858
9447
7499

8. Multiply 976 by 87.
9. Divide 2782542 by 679.
10. Multiply 5489 by 9876.
11. Divide 5099941 by 749.
12. Multiply 876 by 79.
13. Divide 62693256 by 859.
14. Multiply 96879 by 896.

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[2] For full bibliographical reference, see page 169.

ARITHMETIC—Test No. 3. Speed Test—Multiplication

Name ________________ *School* ____________ *Grade* ____

Write on this paper, in the space between the lines, the answers to as many of these multiplication examples as possible in the time allowed.

3 4 9 0 5	4 2 7 4 9	9 5 4 7 6	2 3 9 0 7
2 7 8 2 6	1 9 6 0 5	1 2 8 0 5	1 3 6 5 4
— — — — —	— — — — —	— — — — —	— — — — —
1 2 8 1 5	1 2 7 0 8	2 5 6 0 7	8 2 7 5 4
9 5 7 1 3	6 8 7 6 3	3 5 9 8 3	1 6 9 0 6
— — — — —	— — — — —	— — — — —	— — — — —

ARITHMETIC—Test No. 4. Speed Test—Division

Name ________________ *School* ____________ *Grade* ____

Write on this paper, in the space between the lines, the answers to as many of these division examples as possible in time allowed.

1)8 5)30 8)72 1)0 9)36 2)6 4)24 7)63 6)0 8)32 9)9 3)21 6)48 1)1

5)10 3)9 4)32 6)36 2)0 7)28 5)5 4)36 9)54 8)0 4)12 1)5 2)16 8)48

1)2 9)27 1)4 5)35 9)45 2)2 3)12 8)8 4)28 5)40 2)2 8)16

ARITHMETIC—Test No. 5. Speed Test—Copying Figures

Name ________________ *School* ____________ *Grade* ____

Copy on this paper, in the space between the lines, as many of the printed figures as possible in the time allowed. Write as rapidly as possible, but form the figures as carefully as in working examples.

24967 42976 62947 72964 24976 42967 62974 72946

26974 46927 64972 74926 26947 46972 64927 74962

ARITHMETIC—Test No. 6. Speed Test—Reasoning

Name ________________ *School* ____________ *Grade* _____

Do not work the following examples. Read each example through, make up your mind what operation you would use if you were going to work it, then write the name of the operation selected in the blank space after the example. Use the following abbreviations: "Add." for addition, "Sub." for subtraction, "Mul." for multiplication, and "Div." for division.

	Operation	
1. The children of a school gave a sleigh-ride party. There were 9 sleighs used, and each sleigh held 30 children. How many children were there in the party?......		
2. Two school-girls played a number game. The score of the girl that lost was 57 points and she was beaten by 16 points. What was the score of the girl that won?.....		
3. A girl counted the automobiles that passed a school. The total was 60 in two hours. If the girl saw 27 pass the first hour how many did she see the second?....		
5. A teacher weighed all the children in a certain grade. One girl weighed 70 pounds. Her older sister was 49 pounds heavier. How many pounds did the sister weigh?		
6. A club of boys sent their treasurer to a store to buy baseballs. If they expected him to buy 7 balls at 45 cents each, how much money did they give him to spend?		
9. A girl was five times as strong as her small sister. If the stronger girl was able to lift a weight of 100 pounds, how large a weight could the smaller girl lift?		
16. Five boys gathered nuts which they put into one large pile. Out of this they made five small piles of 197 nuts each. How many nuts were there in the large pile?		

ARITHMETIC—Test No. 7. Fundamentals

Name ________________ *School* ______________ *Grade* ____

In the blank space below, work as many of these examples as possible in the time allowed. Work them in order as numbered, writing each answer in the "answer" column before commencing a new example. Do no work on any other paper.

No.	Operation	Example	Answer	Rt.
1	Addition	*a* 25 + 830 + 122 = (Write answer in this column) *b* 232 + 8021 + 703 + 3030 =		
2	Subtraction	*a* 5496 − 163 = ... *b* 943276 − 812102 = ...		
3	Multiplication	2012 × 213 = ...		
4	Division	158664 ÷ 132 = ...		
5	Addition	6134 + 213 + 4800 + 6005 + 3050 + 474 = ...		
6	Subtraction	73210142 − 49676378 = ...		
7 8	Multiplication	46508 × 456 = ...		
9	Division	27217182 ÷ 6 = ...		
10 11	Division	3127102 ÷ 463 = ...		
12 13	Addition	85586 + 69685 + 39397 + 95836 + 37768 + 69666 + 78888 + 54987 = ...		
14	Subtraction	15655431 − 5878675 = ...		
15 16	Multiplication	78965 × 678 = ...		
17	Division	44502486 ÷ 7 = ...		
18 19	Division	5373003 ÷ 769 = ...		

Arithmetic—Test No. 8. Reasoning.

Name ______________ *School* ____________ *Grade* ______

In the blank space below, work as many of the following examples as possible in the time allowed. Work them in order as numbered, entering each answer in the "answer" column before commencing a new example. Do not work on any other paper.

	Answer	
1. A party of children went from a school to a woods to gather nuts. The number found was but 205 so they bought 1955 nuts more from a farmer. The nuts were shared equally by the children and each received 45. How many children were there in the party?..........		
3. A girl found by careful counting that there were 87 letters more on a page of her history than on a page of her reader. In the first 29 days of school she read 31 pages in each book. How many more letters each day did she read in one book than in the other?............		
5. A girl's record in spelling for 5 days was 19, 18, 20, 16, and 20 words spelled correctly out of 20. If each of the 16 children in the grade had had the same record, what would have been the total number of words spelled correctly by that grade in 5 days?....................		
7. A teacher corrected 2400 arithmetic test papers, 2295 of these he marked "poor," "fair," "good," etc. All the others were marked "unsatisfactory." If each of the papers in this group had 47 mistakes, what was the total number of mistakes in the unsatisfactory papers?......		
8. In two schools five teachers recorded the number of blocks the children walked in going to and from school. The total for one school was 3000 blocks; for the other 2400. The number of children in both schools was 216. How many blocks did each child walk a day?..........		
Total		

COURTIS' TESTS IN FUNDAMENTAL OPERATIONS[4]

Score
No. Attempted.
No. Right .....

SERIES B

Arithmetic—Test No. 1. Addition

Form 2

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

127	996	237	386	180	474	877	537
375	320	949	463	775	787	845	685
953	778	480	827	684	591	981	452
333	886	987	240	260	106	693	904
325	913	354	616	372	869	184	511
911	164	600	261	846	451	772	988
554	897	744	755	505	330	749	559
167	972	195	833	254	820	256	127
554	119	234	959	137	533	258	323

Arithmetic—Test No. 2. Subtraction

Form 1

You will be given four minutes to find the answers to as many of these subtraction examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

107795491 77197029	75088824 57406394	91500053 19901563	87939983 72207316
160620971 80361837	51274387 25842708	117359208 36955523	47222970 17504943

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[4] For full reference, see page 170. Each test in Series B is abbreviated by about 60%.

Arithmetic—Test No. 3. Multiplication

Form 1

You will be given six minutes to work as many of these multiplication examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

$\begin{array}{r}8246\\ \underline{29}\end{array}$	$\begin{array}{r}3597\\ \underline{73}\end{array}$	$\begin{array}{r}5739\\ \underline{85}\end{array}$	$\begin{array}{r}2648\\ \underline{46}\end{array}$	$\begin{array}{r}9537\\ \underline{92}\end{array}$
$\begin{array}{r}4268\\ \underline{37}\end{array}$	$\begin{array}{r}7593\\ \underline{640}\end{array}$	$\begin{array}{r}6428\\ \underline{58}\end{array}$	$\begin{array}{r}8563\\ \underline{207}\end{array}$	$\begin{array}{r}2947\\ \underline{63}\end{array}$
$\begin{array}{r}6385\\ \underline{48}\end{array}$	$\begin{array}{r}8736\\ \underline{502}\end{array}$	$\begin{array}{r}5942\\ \underline{39}\end{array}$	$\begin{array}{r}6837\\ \underline{680}\end{array}$	$\begin{array}{r}4952\\ \underline{47}\end{array}$

Arithmetic—Test No. 4. Division

Form 1

You will be given eight minutes to work as many of these division examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

$25\overline{)6775}$	$94\overline{)85352}$	$37\overline{)9990}$	$86\overline{)80066}$
$73\overline{)58765}$	$49\overline{)31409}$	$68\overline{)43520}$	$52\overline{)44252}$
$68\overline{)39508}$	$49\overline{)28420}$	$52\overline{)21112}$	$73\overline{)33653}$
$28\overline{)23548}$	$54\overline{)48708}$	$39\overline{)32760}$	$67\overline{)61707}$

Woody Scales, Series A[5]

Addition Scale

Name..
When is your next birthday?........How old will you be?.........
Are you a boy or girl?..............In what grade are you?.......

(1)
2
3

(2)
2
3
4

(3)
17
2

(4)
53
45

(5)
72
26

(6)
60
37

(7)
3 + 1 =

(8)
2 + 5 + 1 =

(9)
20
10
2
20
35

(10)
21
33
35

(11)
32
59
17

(12)
43
1
2
13

(13)
23
25
16

(14)
25 + 42 =

(15)
100
33
45
201
46

(16)
0
24
12
15
19

(17)
199
194
295
156

(18)
2563
1387
4954
2065

(19)
$ .75
1.25
.49

(20)
$12.50
16.75
15.75

(21)
$8.00
5.75
2.33
4.16
.94
6.32

(22)
547
197
685
678
456
393
525
240
152

(23)
⅓+⅓=

(24)
4.0125
1.5007
4.10
8.673

(25)
⅜+⅝+⅞+⅛=

(26)
12½
62½
12½
87½

(27)
⅛+¼+½=

(28)
¾+¼=

(29)
4¾
2¼
5¼

(30)
2½
6⅜
3¾

(31)
113.46
49.6097
19.9
9.87
.0086
18.253
6.04

(32)
¾+½+¼=

(33)
.49
.28
.63
.95
1.69
.22
.33

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[5] For full bibliographical reference, see page 171.

	(34)	*(35)*	*(36)*	*(37)*
.36	$\frac{1}{6}+\frac{3}{8}=$	2 ft. 6 in.	2 yr. 5 mo.	$16\frac{1}{3}$
1.01		3 ft. 5 in.	3 yr. 6 mo.	$12\frac{1}{8}$
.56		4 ft. 9 in.	4 yr. 9 mo.	$21\frac{1}{2}$
.88			5 yr. 2 mo.	$32\frac{3}{4}$
.75			6 yr. 7 mo.	
.56				
1.10				
.18				
.56				

(38)
25.091+100.4+25+98.28+19.3614=

Subtraction Scale

Name..

When is your next birthday?........How old will you be?.........

Are you a boy or girl?.............In what grade are you?.......

(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
8	6	2	9	4	11	13	59	78	7−4=	76
5	0	1	3	4	7	8	12	37		60

(12)	*(13)*	*(14)*	*(15)*	*(16)*	*(17)*	*(18)*	*(19)*	*(20)*
27	16	50	21	270	393	1000	567482	$2\frac{3}{4}-1=$
3	9	25	9	190	178	537	106493	

(21)	*(22)*	*(23)*	*(24)*	*(25)*	*(26)*
10.00	$3\frac{1}{2}-\frac{1}{2}=$	80836465	$8\frac{7}{8}$	27	4 yds. 1 ft. 6 in.
3.49		49178036	$5\frac{3}{4}$	$12\frac{5}{8}$	2 yds. 2 ft. 3 in.

(27)	*(28)*	*(29)*	*(30)*
5 yds. 1 ft. 4 in.	10−6.25=	$75\frac{3}{4}$	9.8063−9.019=
2 yds. 2 ft. 8 in.		$52\frac{1}{4}$	

(31)	*(32)*	*(33)*	*(34)*	*(35)*
7.3−3.00081=	1912 6 mo. 8 da.	$\frac{5}{12}-\frac{2}{10}=$	$6\frac{1}{8}$	$3\frac{7}{8}-1\frac{5}{8}=$
	1910 7 mo. 15 da.		$2\frac{7}{8}$	

Multiplication Scale

Name...
When is your next birthday?........How old will you be?.........
Are you a boy or girl?..............In what grade are you?........

(1) $3 \times 7 =$

(2) $5 \times 1 =$

(3) $2 \times 3 =$

(4) $4 \times 8 =$

(5) $\begin{array}{r} 23 \\ \underline{3} \end{array}$

(6) $\begin{array}{r} 310 \\ \underline{4} \end{array}$

(7) $7 \times 9 =$

(8) $\begin{array}{r} 60 \\ \underline{3} \end{array}$

(9) $\begin{array}{r} 254 \\ \underline{6} \end{array}$

(10) $\begin{array}{r} 923 \\ \underline{7} \end{array}$

(11) $\begin{array}{r} 1036 \\ \underline{8} \end{array}$

(12) $\begin{array}{r} 5096 \\ \underline{6} \end{array}$

(13) $\begin{array}{r} 8754 \\ \underline{8} \end{array}$

(14) $\begin{array}{r} 165 \\ \underline{40} \end{array}$

(15) $\begin{array}{r} 235 \\ \underline{23} \end{array}$

(16) $\begin{array}{r} 7898 \\ \underline{9} \end{array}$

(17) $\begin{array}{r} 145 \\ \underline{206} \end{array}$

(18) $\begin{array}{r} 24 \\ \underline{234} \end{array}$

(19) $\begin{array}{r} 9.6 \\ \underline{4} \end{array}$

(20) $\begin{array}{r} 287 \\ \underline{.05} \end{array}$

(21) $\begin{array}{r} 24 \\ \underline{2\frac{1}{2}} \end{array}$

(22) $8 \times 5\frac{3}{4} =$

(23) $1\frac{1}{4} \times 8 =$

(24) $\begin{array}{r} 16 \\ \underline{2\frac{5}{8}} \end{array}$

(25) $\frac{7}{8} \times \frac{3}{4} =$

(26) $\begin{array}{r} 9742 \\ \underline{59} \end{array}$

(27) $\begin{array}{r} 6.25 \\ \underline{3.2} \end{array}$

(28) $\begin{array}{r} .0123 \\ \underline{9.8} \end{array}$

(29) $\frac{1}{8} \times 2 =$

(30) $\begin{array}{r} 2.49 \\ \underline{36} \end{array}$

(31) $\frac{13}{25} \times \frac{15}{32} =$

(32) $\begin{array}{r} \text{6 dollars 49 cents} \\ \underline{8} \end{array}$

(33) $2\frac{1}{2} \times 3\frac{1}{2} =$

(34) $\frac{1}{2} \times \frac{1}{2} =$

(35) $\begin{array}{r} 987\frac{3}{4} \\ \underline{25} \end{array}$

(36) $\begin{array}{r} \text{2 ft. 5 in.} \\ \underline{5} \end{array}$

(37) $2\frac{1}{4} \times 4\frac{1}{2} \times 1\frac{1}{2} =$

(38) $\begin{array}{r} .0963\frac{1}{8} \\ \underline{.084} \end{array}$

(39) $\begin{array}{r} \text{8 ft. } 9\frac{1}{2}\text{ in.} \\ \underline{9} \end{array}$

Division Scale

Name...
When is your next birthday?........How old will you be?.........
Are you a boy or girl?..............In what grade are you?.......

(1) $3\overline{)6}$

(2) $9\overline{)27}$

(3) $4\overline{)28}$

(4) $1\overline{)5}$

(5) $9\overline{)36}$

(6) $3\overline{)39}$

(7) $4 + 2 =$

(8) $9\overline{)0}$

(9) $1\overline{)1}$

(10) $6 \times .. = 30$

(11) $2\overline{)13}$

(12) $2 + 2 =$

(13) $4\overline{)24\text{ lbs. }8\text{ oz.}}$ (14) $8\overline{)5856}$ (15) $\frac{1}{4}$ of 128 = (16) $68\overline{)2108}$ (17) 50 ÷ 7 =

(18) $13\overline{)65065}$ (19) 248 ÷ 7 = (20) $2.1\overline{)25.2}$ (21) $15\overline{)9750}$ (22) $2\overline{)13.50}$

(23) $23\overline{)469}$ (24) $75\overline{)2250300}$ (25) $2400\overline{)504000}$ (26) $12\overline{)2.76}$

(27) $\frac{7}{8}$ of 624 = (28) $.003\overline{).0936}$ (29) $3\frac{1}{2} \div 9 =$ (30) $\frac{3}{4} \div 5 =$

(31) $\frac{5}{4} \div \frac{3}{5} =$ (32) $9\frac{5}{8} \div 3\frac{3}{4} =$ (33) $52\overline{)3756}$

(34) $62.50 \div 1\frac{1}{4}$ (35) $531\overline{)37722}$ (36) $9\overline{)69\text{ lbs. }9\text{ oz}}$

MIXED FUNDAMENTALS

A single scale for measuring ability in all four fundamental operations of whole numbers consists of thirty-five examples. Form I selects its thirty-five examples as follows:

10 examples from the addition scale: nos. 1, 3, 7, 13, 15, 20, 22, 25, 33, 38.
7 examples from the subtraction scale: nos. 3, 7, 13, 17, 20, 25, 31.
10 examples from the multiplication scale: nos. 3, 5, 9, 12, 16, 20, 26, 29, 35, 38.
8 examples from the division scale: nos. 1, 7, 11, 15, 19, 27, 30, 36.

There are additional forms of the Mixed Fundamental Scale made up in similar manner.

CLEVELAND SURVEY ARITHMETIC TESTS [6]

This survey or inventory test consists of fifteen sets. The scope of each set is indicated below:

Sets A, B, C, and D, similar to Tests 1, 2, 3, 4 of the original Courtis Tests, are concerned with combinations in addition, subtraction, multiplication, and division.

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[6] For full bibliographical reference, see page 171.

Set E consists of sixteen examples in single-column addition of five addends. Set F is made up of twenty examples in subtraction of the type of $\begin{array}{r}616\\456\\\hline\end{array}$, $\begin{array}{r}1236\\908\\\hline\end{array}$, and $\begin{array}{r}854\\286\\\hline\end{array}$. Set G consists of 20 examples in which a four-place multiplicand is multiplied by a digit; Set I, of 12 examples in division like $4\overline{)55424}$, $9\overline{)75933}$, and $4\overline{)38068}$.

Sets J, K, L, and N are tests in the four fundamental operations requiring skill of a distinctly higher order. Thus, in addition, there are fourteen examples in single column addition, each having thirteen addends; in division and in multiplication the divisors and the multipliers are of two places. In Set M five addends of four places each are introduced in twelve examples.

Sets H and O are devoted to common fractions. In the former there are twenty-four examples in addition and subtraction, but always with similar denominators; in the latter, twelve examples introduce all the fundamental operations with dissimilar denominators.

MONROE'S DIAGNOSTIC TESTS IN ARITHMETIC[7]

Operations with Integers

Below is given one section of Monroe's Diagnostic Test. The reader is advised to solve each of the examples and thus identify each of the common errors this test is designed to reveal. The other parts of the tests are concerned with the other fundamental operations. Test 10, reproduced below, is allowed two minutes.

Test 10—Multiplication

Attempts......
Rights........

$$\begin{array}{r}560\\37\\\hline\end{array}\qquad\begin{array}{r}807\\59\\\hline\end{array}\qquad\begin{array}{r}617\\508\\\hline\end{array}\qquad\begin{array}{r}840\\80\\\hline\end{array}\qquad\begin{array}{r}730\\96\\\hline\end{array}\qquad\begin{array}{r}609\\70\\\hline\end{array}$$

$$\begin{array}{r}435\\308\\\hline\end{array}\qquad\begin{array}{r}790\\60\\\hline\end{array}\qquad\begin{array}{r}940\\38\\\hline\end{array}\qquad\begin{array}{r}307\\42\\\hline\end{array}\qquad\begin{array}{r}682\\409\\\hline\end{array}\qquad\begin{array}{r}870\\40\\\hline\end{array}$$

$$\begin{array}{r}780\\56\\\hline\end{array}\qquad\begin{array}{r}502\\68\\\hline\end{array}\qquad\begin{array}{r}386\\207\\\hline\end{array}\qquad\begin{array}{r}160\\90\\\hline\end{array}\qquad\begin{array}{r}850\\72\\\hline\end{array}\qquad\begin{array}{r}401\\80\\\hline\end{array}$$

$$\begin{array}{r}817\\109\\\hline\end{array}\qquad\begin{array}{r}400\\30\\\hline\end{array}\qquad\begin{array}{r}730\\52\\\hline\end{array}\qquad\begin{array}{r}605\\84\\\hline\end{array}\qquad\begin{array}{r}392\\306\\\hline\end{array}\qquad\begin{array}{r}590\\30\\\hline\end{array}$$

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[7] For full bibliographical reference, see page 172.

MONROE'S STANDARDIZED REASONING TEST IN ARITHMETIC[8]

For Grades 6 and 7.—Test II, Form I.

This reasoning test in arithmetic is made up of fifteen problems. Each is allowed credit for correctness of principle (P) and correctness of answer (C). These are indicated with each problem. Without the norms supplied by the author of the test, the score of a pupil has relatively little significance.

1. A girl having ¾ yd. of ribbon, bought ⅛ yd. more. What part of a yard had she then? .... P, 2; C, 1.
2. A piece of ribbon 4⅝ yards long is cut from a bolt containing 10 yards. How many yards are left? .. P, 2; C, 2.
3. There are 31.5 gallons in a barrel. How many gallons are there in 63 barrels? .... P, 2; C, 2.
4. If a horse eats ⅜ bu. of oats a day, how long will 6 bu. last? .... P, 3; C, 2.
5. When a 20-pound cheese is worth $1.90, how much will a 10-pound cheese cost? .... P, 1; C, 1.
6. Four loads of hay are to be put into a barn. The first load weighs 1.125 tons; the second, 1.75 tons; the third, 1.8 tons; the fourth 1.9 tons. Find the weight of the four loads. .... P, 1; C, 2.
7. A baker used ⅝ lb. of flour to a loaf of bread. How many loaves could he make from a barrel (196 lbs.) of flour? .... P, 3; C, 2.
8. My telephone bill is $12.85 a month. At that rate how much should I pay in 2¾ years? .... P, 2; C, 2.
9. A man spends $6.50 for board, $12.25 for clothing, $5.20 for books, and had $12 left. How many dollars and cents had he at first? .... P, 1; C, 2.
10. A boy saves 1¾ cents on a picture by doing his own developing and printing. This makes a saving of how much on each dozen pictures? .... P, 2; C, 1.
11. Make out the following account for a day. Cash on hand, $174.30; Receipts, mdse., $12.50; $6.75, $0.42, $17.30, $9.50, $42.75; Expenses, Perry and Co., bill, $75.82. .... P, 3; C, 3.
12. Muslin is to be bought for 12 new curtains each requiring 2⅞ yds. How much will the muslin cost at 12½ cents a yard. .... P, 3; C, 3.

---

[8] For full bibliographical reference, see page 172.

13. A market man has 7850 pounds of ice put into his refrigerator at one time. How much does it cost at $3.90 a ton? .... P, 2; C, 3.

14. A man bought two suits of clothes one costing $35.75 and the other $28.50. How much more did the one cost than the other? .... P, 1; C, 1.

15. A farmer raised 500 bushels of wheat on a field of 40 acres. What was the average yield per acre?...... P, 2; C, 2.

Teacher's Diagnosis
for Pupil ________

DIAGNOSTIC CHART
FOR
INDIVIDUAL DIFFICULTIES

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Bloomington, Illinois
Printed in U. S. A.

FUNDAMENTAL PROCESSES IN ARITHMETIC

Prepared by G. T. Buswell and Lenore John

Name ________School ________Grade ______Age ______IQ ______

Date of Diagnosis:__________Add___; Subt.___; Mult.___· Div.___

Teacher's preliminary diagnosis________________________

Division: (Place a check before each habit observed in the pupil's work)

___ d1 Errors in division combinations
___ d2 Errors in subtraction
___ d3 Errors in multiplication
___ d4 Used remainder larger than divisor
___ d5 Found quotient by trial multiplication
___ d6 Neglected to use remainder within problem
___ d7 Omitted zero resulting from another digit
___ d8 Used wrong operation
___ d9 Omitted digit in dividend
___d10 Counted to get quotient
___d11 Repeated part of multiplication table
___d12 Used short division form for long division
___d13 Wrote remainders within problem
___d14 Omitted zero resulting from zero in dividend
___d15 Omitted final remainder
___d16 Used long division form for short division
___d17 Counted in subtracting
___d18 Used too large a product
___d19 Said example backwards
___d20 Used remainder without new dividend figure
___d21 Derived unknown combination from known one
___d22 Had right answer, used wrong one
___d23 Grouped too many digits in dividend
___d24 Error in reading
___d25 Used dividend or divisor as quotient
___d26 Found quotient by adding
___d27 Reversed dividend and divisor
___d28 Used digits of divisor separately
___d29 Wrote all remainders at end of problem
___d30 Misinterpreted table
___d31 Used digit in dividend twice
___d32 Used second digit of divisor to find quotient
___d33 Began dividing at units digit of dividend
___d34 Split dividend
___d35 Used endings to find quotient

DIVIDE:

(1) 2)8 6)54	(2) 4)1684 3)1869	(3) 3)693 2)46
(4) 5)5 12)12	(5) 3)18300 2)8040	(6) 3)1209 8)5608
(7) 8 ÷ 1 = 6 ÷ 2 =	(8) 2)9 5)48	(9) 23)483 32)992
(10) 3)1000 9)80000	(11) 2)904 7)31542	(12) 2)6140 6)54036
(13) 9)41670 3)17385	(14) 16)576 25)925	(15) 21)1491 34)16354

(16) 5)35156 8)100600	(17) 5)151 6)204183
(18) 17)359 78)62644	(19) 37)15133 46)231104
(20) 400)295600 530)332840	(21) 924)680272 395)302175

Similar diagnostic analyses are provided for the other fundamental operations.

Sangren-Reidy Instructional Tests in Arithmetic[9]

This series of tests identifies the major unit skills in each arithmetical process as an aid in developing arithmetical ability. For each grade, the authors offer a series of tests, one test for each month. Each test is designed to reveal, "The particular types of difficulty of the individual pupil" in both fundamental operations and problem-solving.

The following specimens present the framework around which a test is constructed.

Grade III—Test I

Unit of Skill	*Description of Skill or Ability*	*Example*
A	Ability to add single column (keeping one's place in the column): Three or more digits........................	3 2 7
B	Ability to identify columns........................	22 23
C	Ability to add with carrying one........................	32 59
D	Ability to add with carrying two........................	39 29 45
	Ability to write in correct position the numbers which should be added:	
E	Sum less than 10........................	3, 4, 2
F	Greater than 10 and written........................	5, 4, 6, 2
G	Sum includes zero........................	60, 20, 40

---

[9] For bibliography reference, see page 172.

Grade V—Test II

Unit of Skill	*Description of Skill or Ability*	*Example*
	As to procedure:	
	Proper fractions	
A	When one given denominator is common denominator........................	¼ + ½ =
B	Addend a whole number and preceding addend is a fraction........................	⅓ + 3 =
	Addend a proper fraction and preceding addend a mixed number	
C	When one denominator is common denominator................................	2¼ + ½ =
D	When denominators have common factor only................................	2⅙ + ⅙ =
E	One addend a mixed number and preceding addend a whole number..................	4 + 2⅜ =
F	Vice versa of the above....................	1⅛ + 5 =
	Three or more fractions or mixed numbers added in a column	
G	Dissimilar fractions........................	2¼ 3⅓ 5⅙
H	Analysis of answers—irreducible..............	¾

Grade V—Test XI

Unit of Skill	*Description of Skill or Ability*	*Example*
	Ability to read and write decimals and decimal mixed numbers:	
A	Thousandths	seven hundred thirty-two thousandths Write .623 in words
B	To change a common fraction to tenths	3/10
C	To change a common fraction to hundredths	3/100
D	Change decimal to fraction	.3
E	Change mixed number to fraction	1.36
F	Ability to recognize and use the decimal forms of important fractions	½; ⅔; ¾
	Ability to write decimals:	
G	Fraction, in words	sixteen hundredths
H	Mixed decimal	seven and sixteen hundredths
I	Number of zeros necessary	seventy-one and three hundredths
J	Ability to recognize differences in notation of decimals	.1 tenths (1.3) .01 hundredths (6.32) .001 thousandths (9.124)

Grade VI—Test X

Unit of Skill	*Description of Skill or Ability*	*Example*
A	Ability to change fractions to percents: Mixed numbers	1⅝ =
B	Discount: Percentage Case II	A suit is marked down from $35 to $25. What percent is it marked down?
C	Ability to understand commission	A boy sells 35 magazines at 25c. each and gets 10% commission. What commission does he make?
D	Profit and Loss: Percentage Case I	John's father bought a car for $1350. He sold it later at a 25% loss. What was the loss in money?
E	Percentage Case II	I bought a suit for $60 and later sold it for $40. What percent of the cost is the selling price?
F	Understanding interest: Case I	Find interest for 1 yr. on $200 at 6%.
G	Using fractions of a year	8 mo. on $100 at 6½% =

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